DEPARTMENT OF THE NAVY

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DDS9110-7

DESIGN OF FOUNDATIONS AND OTHER STRUCTURES TO RESIST SHOCK LOADINGS

CONTENTS

9110-7-a. Nomenclature and symbols; abbreviations.

Symbol	Nomenclature	Unit
A, B, C, ...	points of structure	–
A	area of cross section	in.^{2}
A	design acceleration	g's
a	acceleration	in./sec.^{2}
a, b, c, ...	matrix elements	–
b	breadth of section	in.
c	distance from extreme fiber to neutral axis	in.
c	damping constant	k.-sec./in.
c.p.s.	cycles per second	–
D	divisor for normalization; final value = $g\Delta/\delta\omega^2$	–
d	depth of section	in.
d	deflection	in.
E	modulus of elasticity (29,600 for steel)	k.s.i.
F_C	column strength (See DDS9110-4.)	k.s.i.
F_Y	yield strength	k.s.i.
f	stress; axial stress	k.s.i.
f_B	bending stress	k.s.i.
f_S	shear stress	k.s.i.
F.S.	factor of safety	–
G	modulus of elasticity in shear (11,400 for steel)	k.s.i.
g	acceleration of gravity (386)	in./sec.^{2}

Symbol	Nomenclature	Unit
h	time interval	sec.
h	height from base to center of gravity	in.
HTS	high-tensile steel	–
I	moment of inertia of cross section	$in.^4$
K	spring constant (stiffness)	k./in.
K_E	structural parameter (effective volume)	$in.^3$
K_Q	required structural parameter = Q/q	$in.^3$
k.	kip (1000 lbs.)	–
k.s.i.	kips per square inch	–
L	length of member	in.
$\bar{L}$	length effective in absorbing energy	in.
m	mass (= W/g)	$k.\text{-}sec.^2/in.$
M	bending moment	in.-k.
m	moment per unit load	in.
MS	medium steel	–
msec.	millisecond (0.001 sec.)	–
n	rigid-base natural frequency of vibration	c.p.s.
P	load; design load	k.
p	force on member per unit load	–
Q	design energy; internal energy of structure	in.-k.
q	energy in unit volume of material at yield stress	$k./in.^2$
R	reaction	k.
r	radius of gyration of cross section	in.
S	statical moment of area about neutral axis	$in.^3$
T	duration of load or motion	sec.
t	time	sec. or msec.
u	displacement; amplitude	in.
V	design velocity; step velocity change	in./sec.
V	shearing force	k.
v	velocity	in./sec.
v	shear per unit load	–
W	weight	k.
$\bar{W}$	modal effective weight = $(\Sigma Wu)^2/\Sigma Wu^2$	k.
WF	wide-flange beam section (Preceding and following numbers indicate depth, in inches, and unit weight, in lbs. per linear foot, respectively.)	–
Z	section modulus	$in.^3$
Δ	influence coefficient times a constant	–
δ	influence coefficient (deflection from unit load)	in./k.
Σ	indicates summation	–
ϕ	angular deformation of beam	radian
ω	natural circular frequency (= $2\pi n$)	rad./sec.

Subscripts. The following subscripts are used, in addition to those whose meanings are indicated above:

Symbol	Nomenclature
A, B, C, ...	panel points, load points
AB, BC, etc.	These indicate that values are applicable between points designated.
a, b, c, ...	columns eliminated from matrix for 2nd, 3rd, 4th, ... modes
B	base, to which foundation is attached
C	from damping force
E	value at yield stress
i 1, 2, 3, ... n	row of matrix
j 1, 2, 3, ... n	column of matrix
K	from spring force
o	initial value
P	plastic
W	web of beam
x	longitudinal (fore-and-aft) direction
y	athwartship direction
z	vertical direction
0	limit-design value
1, 2, 3, ...	modes of vibration (second subscript when first indicates deflection)
1, 2, 3, ... n	deflections of masses (centers of gravity of lumped masses); corresponding load points (with load in direction of deflection)
1-1, 1-2, etc.	These indicate deflections and load points for influence coefficients, which may be interchanged, in accordance with Maxwell's theorem.

9110-7-b. Introduction.

The nature of the loading. – Under explosive loading, the shell of a ship is subjected to a very high pressure for a brief period of time. This imparts high accelerations to the loaded parts. As the shell moves, the ship's structure is deformed and transmits forces to the remainder of the hull and to masses attached to it. Inertia forces oppose the motion of the hull. Ultimately, the ship is brought to rest by gravity forces and hydrodynamic resistances. Figure 1 shows typical velocity-time curves for two locations on a destroyer bulkhead during a full-scale shock test.

Response of structure. – If a foundation is sufficiently rigid, it will bring the mass it supports to the same velocity as the base (hull structure to which the foundation is attached) within the period during which the latter is accelerating. For a constant base acceleration, applied instantaneously, the maximum acceleration of an elastically supported mass would be twice that of the base. With a more flexible foundation, however, the natural period of vibration may be much greater than the interval of base acceleration. In this case, relative movement between base and mass limits the acceleration applied to the latter. A step change in base velocity would give the mass a relative velocity of equal magnitude. These two idealized types of motion, shown by figure 2, provide a basis for empirical design criteria.

Design criteria. – For design of foundations and similar structures, shock inputs may be specified in terms of the following:

A static equivalent acceleration (A), or design load (P).

A step change in velocity (V), or the energy associated therewith (Q). Only the lesser of the two requirements need be satisfied.

The values of the inputs used for design must be obtained from applicable specifications. They will depend upon the type of ship, weight of equipment, location within the ship, the service required, and direction of loading. The criteria used herein are to be considered only as illustrative.

In general, the design loads or energies are associated with yield strength (F_Y), column strength (F_c), ultimate compressive strength of plating, or similar limiting value; that is, no factor of safety (F.S.) is applied to the structure as a whole. For individual parts of the structure, however, factors of safety may be specified.

Shocks of relatively low intensity should be resisted without permanent deformation. For greater intensities, yielding may be permissible if accurate alinement need not be preserved. Under extreme conditions, deformation is inevitable, but it is important that items of equipment be prevented from tearing loose and becoming missiles. If connections are adequate and materials are sufficiently ductile, shocks of greater than design intensity will produce permanent deformation, but not complete structural failure as would an excessive static load. Beams, especially, are able to survive shocks far greater than those which just give yield stress. And, although columns are more likely to be weakened by deformation, the shock accelerations may end before complete failure can occur. For these reasons, the empirical design values may be much lower than the accelerations and velocities measured during shock conditions.

Applicability of this design data sheet. – The design procedure outlined in 9110-7-c is applicable to a foundation supporting equipment which is to be treated as a rigid mass, when criteria are stated in terms of accelerations or loads, and energies or velocities. Deflection is assumed to be essentially in the direction of the shock motion.

When criteria are similarly specified but the motion involves several deflection components simultaneously, or more than one mass, 9110-7-d applies. An example illustrates calculation of mode shapes and frequencies of a multi-mass system. The normal-mode method of NAVSHIPS 250-423-30 is then used to compute shock responses.

9110-7-e shows how loadings, applicable to a single-mass system, can be more accurately defined by means of shock spectra.

If the mass is supported in such a way that its displacement relative to the base can be very large, it is necessary to examine the time history of the motion to determine forces and deformations. This applies to shock mitigating devices, but not to foundations generally. 9110-7-f gives an approximate method which may be used to establish spectra for various motions or forces, or to analyze more complex systems, or those whose response cannot be approximated by the load-deflection curve of figure 3.

With very soft mountings, forces other than shock loadings often govern design. For example, in a seaway the ship has comparatively low accelerations, but amplitudes preclude their attenuation by deformation of supports.

9110-7-c. Design of Single-Mass Systems Based on Load and Energy Criteria.

Load criteria. – If acceleration governs, design for shock loadings is no different from design for static loadings. In fact, many loads other than those from underwater explosion are similarly reduced, either explicitly or empirically, to static equivalents. For simple structures, the design procedure is then comparatively direct. Required section moduli can be calculated for beams, and sectional areas for axially loaded members. The latter must sometimes, of course, be revised to suit slenderness ratios, which are unknown until specific sections are selected. Bending moments may depend on relative stiffnesses, but by using limit design principles this difficulty is minimized.

Energy criteria. – If a structure is large, particularly if high-strength materials are used, the energy (or velocity) criteria may permit design for smaller loads. For a simple structure, a direct design procedure to obtain the required volume of material may then be applicable, if the approximate proportions and patterns of stresses are predictable. For instance, if the foundation consists essentially of beams of known length, an "I" section can be chosen to satisfy an energy criterion almost as readily as one having the section modulus corresponding to a given load. If the structure is more complex, the most feasible procedure may be to design for an assumed load, calculate the energy which this structure can absorb, and then revise as necessary. The assumed load need never be greater than the specified load criterion.

An energy criterion implies that the internal energy absorbed by the structure is equal to the energy imparted to the mass by giving it a velocity relative to the ship structure. Energy (Q) and velocity are related by the formula

$$Q = \frac{WV^2}{2g},$$

where W is the weight supported and g is the acceleration of gravity. Ordinarily it will be specified that all or a stated portion of this energy be absorbed elastically.

The energy absorbed by a unit volume of material is the product of the strain and the mean stress. Under tensile or compressive stress (f) up to the elastic limit, it is $f^2/2E$. (E is the modulus of elasticity.) For design of simple structures, it is convenient to let "q" be the energy in a unit volume of material at the stress corresponding to yield strength. The required effective volume of material is then $K_Q = Q/q$, where $q = F_Y{}^2/2E$.

If a member is axially loaded to yield strength throughout, its effective volume, K_E, is the actual volume. If not, K_E must be reduced to allow for variation in the value of $(f/F_Y)^2$. At a beam cross section of area, A, where bending stress, f_B, has reached yield at the extreme fibers, integration of $(f_B/F_Y)^2 dA$ gives $(r/c)^2 A$. (r is the radius of gyration and c is the maximum distance from the neutral axis.) The equivalent, uniformly-stressed area is therefore Z/c. (Z is the section modulus.) Figure 4 indicates values of Z/c in terms of area for a number of shapes.

Allowance must also be made for any variations in stresses along the beam. If curves of bending moment (M) and Z/c are smooth for a beam or portion thereof, Simpson's first rule provides a convenient means of evaluating the effective volume. Then, if C is the midpoint of beam, AB (of length, L_{AB}),

$$K_E = \frac{L_{AB}}{6}\left\{\left[\left(\frac{Z}{c}\right)\left(\frac{f_B}{F_Y}\right)^2\right]_A + 4\left[\left(\frac{Z}{c}\right)\left(\frac{f_B}{F_Y}\right)^2\right]_C + \left[\left(\frac{Z}{c}\right)\left(\frac{f_B}{F_Y}\right)^2\right]_B\right\}.$$

As long as M/Z does not exceed yield stress, the internal energy of the beam may be written

$$Q = \frac{L_{AB}}{12E}\left[\frac{M_A{}^2}{I_A} + 4\frac{M_C{}^2}{I_C} + \frac{M_B{}^2}{I_B}\right],$$

where I_A, I_B, and I_C are the moments of inertia of the sections indicated by the subscripts. If the beam has constant cross section and uniformly varying moment, M_C may be replaced by $(M_A + M_B)/2$. Then

$$Q = \frac{L_{AB}}{6EI_{AB}}\left[M_A{}^2 + M_A M_B + M_B{}^2\right].$$

When stressed to yield at A

$$Q_E = \frac{L_{AB} M_A{}^2}{6EI_{AB}}\left[1 + \frac{M_B}{M_A} + \left(\frac{M_B}{M_A}\right)^2\right]; \quad K_E = \left(\frac{L_{AB}}{3}\right)\left(\frac{Z}{c}\right)\left[1 + \frac{M_B}{M_A} + \left(\frac{M_B}{M_A}\right)^2\right].$$

Note that if $M_B = 0$, as on a cantilever, or if $M_B = -M_A$,

$$K_E = \left(\frac{L_{AB}}{3}\right)\left(\frac{Z}{c}\right).$$

That is, effective length, $\bar{L}_{AB} = L_{AB}/3$. $\bar{L}_{AB}$ is minimum $(L_{AB}/4)$ when $M_B = -0.5\,M_A$.

Shear. – A structure comprised of short, deep beams may absorb a considerable amount of energy through shear deformation. A unit volume of elastic material, at stress, f_S, increases Q by $f_S{}^2/2G$. For steel, the modulus in shear, G, is about 11,400 ksi. An allowable limit of 0.6 F_Y is usually specified for f_S.

The shear stress at any point on a cross section is given by the expression, $f_S = VS/Ib$, where V is the total shear, S is the statical moment of area (beyond the point in question) about the neutral axis, I is the moment of inertia of the entire cross section, and b is the breadth of the section at the point. For most beams, however, only an average stress, based on web area, need be calculated. That is

$$f_S = V/A_W .$$

Ordinarily, it need only be established that the bending and shear stresses are within their respective allowable limits. In special cases, however, consideration of combined stresses may be required.

Limit design. – A fundamental principle of limit design is that a structure will not collapse if a pattern of stresses, not exceeding yield, can be found to hold the applied loads in equilibrium. As compared to the purely elastic condition, the application of limit design generally permits a redistribution of bending moments and a redistribution of stresses over the cross sections where plastic hinges are formed.

For the fixed-end beams of Arrangement No. 4 (see figure 5(d)), elastic theory indicates moments at C and D which are only 43 percent of the end moments (see figure 6(b)). However, the formation of plastic hinges at A and B would not lead to collapse of the structure. Slight rotation at these points would permit moments between C and D to increase until the full strength of the beams is developed there also. (Even for a strictly elastic condition, there would probably be enough end rotation to make the equal-moment assumption a realistic one.) With the formation of a third hinge at C or D, the structure would become a mechanism and, under static load, collapse would ensue. Under shock loading, complete collapse would be unlikely, but there might be significant permanent deformation.

As each plastic hinge begins to form, the zone of yield stress spreads from the outermost fibers toward the neutral axis. In the limiting condition (in pure bending), half the area is in tension and half in compression. For an "I" section, bending in its strong direction, the plastic section modulus is only about 14 percent greater than the elastic one, since the flange stress can change but little. For the web alone, or any rectangular section of breadth, b, and depth, d, the increase is from $bd^2/6$ to $bd^2/4$, or 50 percent. For unsymmetrical sections, and those under simultaneous axial loads, plastic moduli may be considerably greater than the elastic values.

In computing the strength of a structure for comparison with load criteria, it will generally be permissible to take advantage of the redistribution of both moments and stresses. However, the normal procedure for computing internal energy is based on the elastic properties of sections, although moment diagrams may be adjusted. Increased strength at moment peaks would permit an increase in the elastic energy stored in a beam, but it should be borne in mind that this criterion is only a rough measure of shock resistance. In the calculations which follow, the aim has been to obtain a reasonably consistent empirical basis for design, without unwarranted refinements, particularly those not routinely considered for other design loadings.

Vibration frequency. – Natural frequency is an important parameter in the consideration of shock loadings. Figure 12, discussed more fully in 9110-7-e, shows the relationship between values calculated for the examples and the criteria used for vertical loadings.

If a load, P, imparts an internal energy, Q, to a linear elastic system, the corresponding external work is P/2 times the deflection. Stiffness, K, is therefore equal to $P^2/2Q$. Hence the natural frequency of a simple system is given by:

$$n = \frac{1}{2\pi}\sqrt{\frac{K}{m}} = \left(\frac{\sqrt{g}}{2\pi\sqrt{2}}\right)\left(\frac{P}{\sqrt{QW}}\right) = 3.13\sqrt{\frac{K}{W}} = \frac{2.21P}{\sqrt{QW}} \text{ c.p.s.}$$

Another form of the frequency formula is $n = 3.13/\sqrt{\delta W}$ c.p.s., where δ is the "influence coefficient," or deflection under unit load. (See 9110-7-d for examples of calculations.) δW is the "static deflection" of the mass.

The foregoing formulas are based on the assumption that the vibratory motion is in the direction of the applied forces, and that the effect of rotary inertia is negligible. It should be noted, also, that actual frequencies tend to be lower than calculated values because the ship structure supporting the foundation is never absolutely rigid. Thus the true values of Q and δ may be increased appreciably by stresses in structure not considered in arriving at the design values. As regards the effects of shock loading on the foundation itself, neglect of the energy stored in the base structure gives an error on the safe side.

Examples. – Foundation structure will be designed to support a weight of 14 k., including allowance for the weight of the foundation itself. (Generally about half of the foundation weight is lumped with that of the equipment.) Figure 5 shows the various structural arrangements considered.

LOAD AND ENERGY CRITERIA

Direction	Design Ratios		Design Values	
	P/W (= A)	Q/W* (= $V^2/2g$)	P - k.	Q - in.-k.
Vertical	40	5.2	P_z = 560	Q_z = 72.8
Athwartship	24	2.6	P_y = 336	Q_y = 36.4
Longitudinal	16	1.3	P_x = 224	Q_x = 18.2

*Corresponding value of P/W to be at least 6.

Arrangement No. 1. – All members are loaded axially. First the legs are designed to suit the vertical load criterion. Elastic energy is then calculated but found to be insufficient to permit a reduction in scantlings. Although the specified athwartship load is less than the vertical, it requires increased strength in one pair of legs. (The use of X-bracing instead of the single diagonals would give a more nearly uniform distribution of stresses.) This time, however, the energy capacity of the structure (Q_{Ey}) exceeds the required value (Q_y). Scantlings are reduced accordingly, and energy is recalculated. Design for the longitudinal criteria would be essentially similar and is omitted. For information, vibration frequencies (n_y and n_z) are computed from the load and energy data.

The equipment is to be supported by a framework, as shown in figure 5(a). Medium steel (MS) will be used, giving F_Y = 33 k.s.i., and q (for the energy criteria) = $(33)^2/(2 \times 29{,}600) = 0.0184$ k./in.^{2}.

Vertical load: If the slenderness ratio is 40 or less, $F_C = F_Y$. The required area for each of four equal legs is then

$$A = P_z/4F_Y = 560/(4 \times 33) = 4.24 \text{ in.}^2$$

Vertical energy:

$$K_{Qz} = Q_z/q = 72.8/0.0184 = 3960 \text{ in.}^3$$

$$K_{Ez} = 4AL = 4 \times 4.24 \times 60 = 1020 \text{ in.}^3 < 3960.$$

Hence the load criterion governs.

Athwartship load: Required area for left leg, AC, is obtained by taking moments about intersection of leg, BD, and diagonal, AD.

$$A_{AC} = P_y(L_{BD} + h)/2F_Y L_{CD} = 336(60 + 20)/(2 \times 33 \times 45) = 9.05 \text{ in.}^2$$

For right leg, BD, take moments about A.

$$A_{BD} = P_y h/2F_Y L_{CD} = 336 \times 20/2970 = 2.26 \text{ in.}^2,$$

which is less than the 4.24 required for the vertical load. For the diagonal, since the legs exert no horizontal components,

$$A_{AD} = P_y L_{AD}/2F_Y L_{CD} = 336 \times 75/2970 = 8.48 \text{ in.}^2$$

Athwartship energy:

$$K_{Qy} = Q_y/q = 36.4/0.0184 = 1980 \text{ in.}^3$$

For scantlings with the minimum areas required for P_z and P_y:

Member	2 × A in.^{2}	L in.	f k.s.i.	$(f/F_Y)^2$	$2AL(f/F_Y)^2$ in.^{3}
Left legs	2 × 9.05	60	33	1.000	1090
Right legs	2 × 4.24	60	17.6	0.284	140
Diagonals	2 × 8.48	75	33	1.000	1270
		Structural parameter:	$K_{Ey} = \Sigma AL(f/F_Y)^2$	=	2500 in.^{3}

Since this exceeds K_{Qy}, scantlings may be reduced. If all members could be changed proportionately, the reduction would be 1980/2500 = 0.79. However, design of the right leg is controlled by the vertical load, and the minimum areas cannot be provided exactly. The somewhat greater scantlings shown by the table below will therefore be used.

Load to give yield stress in leg, AC.

$$P_{Ey} = 2A_{AC} F_Y L_{CD}/(L_{BD} + h) = 2 \times 8.10 \times 33 \times 45/(60 + 20) = 301 \text{ k.}$$

Corresponding stress in leg, BD, is

$$f_{BD} = P_{Ey} h/2A_{BD} L_{CD} = 301 \times 20/(2 \times 4.41 \times 45) = 15.2 \text{ k.s.i.}$$

Stress in diagonal, AD, is

$$f_{AD} = P_{Ey} L_{AD}/2A_{AD} L_{CD} = 301 \times 75/(2 \times 8.10 \times 45) = 31.0 \text{ k.s.i.}$$

$$L/r = 75/1.37 = 55; \; F_C = 31 \text{ k.s.i.}$$

Member	O.D. in.	Thk. in.	2 × A in.^{2}	L in.	f k.s.i.	$(f/F_Y)^2$	$2AL(f/F_Y)^2$ in.^{3}
Left legs	4.5	0.674	2 × 8.10	60	33	1.000	970
Right legs	4.5	0.337	2 × 4.41	60	15.2	[illegible]	110
Diagonals	4.5	0.674	2 × 8.10	75	31	0.882	1070
			Structural parameter:		$K_{Ey} = \Sigma AL(f/F_Y)^2$	=	2150 in.^{3}

Vibration frequencies: In the athwartship direction,

$$P_{Ey} = 301 \text{ k.}, \; Q_{Ey} = K_{Ey}q = 2150 \times 0.0184 = 39.6 \text{ in.-k.}$$

$$n_y = 2.21, \; P_{Ey}/\sqrt{Q_{Ey}W} = 2.21 \times 301/\sqrt{39.6 \times 14} = 28 \text{ c.p.s.}$$

In the vertical direction, frequency can be computed similarly (using values of P and Q which represent the structure, rather than the design criteria), giving $n_z = 89$ c.p.s. An assumption somewhat less conservative than that the vertical load is equally divided among the four legs would be that they are equally stressed in spite of differences in area. In that case, stiffness would be equal to $\Sigma AE/L$. Then

$$K = 2(8.10 + 4.41) \times 29{,}600/60 = 12{,}300 \text{ k./in.},$$

and

$$n_z = 3.13\sqrt{K/W} = 3.13\sqrt{12{,}300/14} = 93 \text{ c.p.s.}$$

Arrangement No. 2. – Arrangement No. 2 consists primarily of a pair of longitudinal girders. Design is based on the vertical load criterion and limit-design section modulus. Energy, in bending and shear, is then calculated. Q_{Ez} is less than Q_z so the beam size may not be changed.

For high-tensile-steel (HTS), $F_Y = 45$ k.s.i. and $q = 0.034$ k./in.^{2}. The girders are simply supported and are considered rigid in way of the equipment.

Vertical load: The required section modulus for each girder is

$$P_z L_{AC}/4F_Y = 560 \times 36/(4 \times 45) = 112 \text{ in.}^3$$

If the limit-design strength is used, an 18 WF 55 gives

$$Z_0 = 1.14\, Z = 1.14 \times 98.2 = 112 \text{ in.}^3$$

Corresponding average shear stress in webs is

$$f_S = P_z/4A_W = 560/4(0.390 \times 16.9) = 21.2 \text{ k.s.i. (which is less than } 0.6\, F_Y).$$

Vertical energy: Bending energy at yield stress is

$$(2/3)(L_{AC} + L_{DB})(Z/c)\, q = (2/3)(36 + 36)(98.2/9.06)(0.034) = 17.7 \text{ in.-k.}$$

Load corresponding to yield stress in bending is

$$P_{Ez} = 4ZF_Y/L_{AC} = 4 \times 98.2 \times 45/36 = 491 \text{ k.}; \quad f_S = 18.6 \text{ k.s.i.}$$

Approximate shear energy is

$$4A_W L_{AC}(f_S^2/2G) = 4 \times 6.6 \times 36 \times 18.6^2/(2 \times 11{,}400) = 14.4 \text{ in.-k.}$$

Total internal energy is then

$$Q_{Ez} = 17.7 + 14.4 = 32.1 \text{ in.-k.}$$

This is less than the required value ($Q_z = 72.8$), so the load criterion governs.

Athwartship and longitudinal criteria: The section modulus of an 18WF55 beam is only 11.1 in.^{3} for bending in the weaker direction. Although the limit-design section modulus is about 1.5 times this, it is evident that the girders, acting independently, would have inadequate strength. The usual design practice is to tie the two girders together by means of cross bracing or deck plating. If this is done, the horizontal loads are not likely to cause stresses as great as those from vertical loading. Overturning moments must be taken into account, but these would be significant only if the center of gravity were high compared to the spacing of the girders. If it is necessary to consider the energy criteria, it may be noted that the ends of the girders would be subject to vertical moments varying from zero at the supports to equal maxima at the equipment. The value of Q_E, due to bending, is therefore the same for all three directions, except that stresses may have to be reduced to suit the horizontal components.

Arrangement No. 3. – This is similar to Arrangement No. 2, but the equipment is supported at four points so that the central parts of the girders can bend. This increases flexibility and energy capacity without affecting strength. For an elastic-plastic design, a smaller value of Q_z permits a reduction in beam scantlings. Both bending and shear effects are calculated, but the importance of the latter decreases as the depth-to-span ratio decreases. An estimate is made of the permanent deformation resulting from the full, elastic-design motion input. To illustrate the calculation of energy under combined axial and bending stresses, P_{Ex} (the longitudinal load corresponding to yield stress) and Q_{Ex} are computed.

The addition to Q_{Ez}, as compared to Arrangement No. 2, is

$$2L_{CD}(Z/c)\,q = 2 \times 48 \times 10.84 \times 0.034 = 35.4 \text{ in.-k.},$$

and

$$Q_{Ez} = 32.1 + 35.4 = 67.5 \text{ in.-k.},$$

which is still less than the given elastic-design value of Q_z. Assume, however, that the specifications require that only 40 percent of Q_z need be absorbed elastically. Then $0.40 \times 72.8 = 29.1 < 67.5$.

Z/c for any WF section, loaded in the strong direction, is about $(2/3)A$ (see figure 4). Required sectional area is therefore reduced, approximately proportionately, from the area of the 18WF55 section.

$$16.19 \times 29.1/67.5 = 7.0 \text{ in.}^2$$

A 12WF27 section gives the following:

In bending,

$$K_{Ez} = 2(Z/c)(L_{AC}/3 + L_{CD} + L_{DB}/3) = 2(34.1/5.98)(12 + 48 + 12) = 820 \text{ in.}^3$$

$$Q_{Ez} = K_{Ez}q = 820 \times 0.034 = 27.9 \text{ in.-k.}$$

$$P_{Ez} = 4ZF_Y/L_{AC} = 4 \times 34.1 \times 45/36 = 170 \text{ k.}$$

In shear, since reactions are equal,

$$f_S = P_{Ez}/4A_W = 170/4(0.240 \times 11.15) = 15.9 \text{ k.s.i.} > 0.6\,F_Y$$

$$Q_{Ez} = 2A_W(L_{AC} + L_{DB})f_S^2/2G$$

$$= 2 \times 2.68(36 + 36)(15.9)^2/22{,}800 = 4.3 \text{ in.-k.}$$

Total $Q_{Ez} = 27.9 + 4.3 = 32.2$ in.-k., about 44 percent of the full, elastic-design value. (Note that for the shallower beam, shear stresses and deflections are of less importance.)

Permanent deformation, under the specified design input, may be approximated as follows:

Energy to be absorbed plastically,

$$Q_{Pz} = Q_z - Q_{Ez} = 72.8 - 32.2 = 40.6 \text{ in.-k.}$$

For each girder,

$$M_0 = 1.14\,ZF_Y = 1.14 \times 34.1 \times 45 = 1750 \text{ in.-k.}$$

Rotation about plastic hinge, at D,

$$\phi = Q_{Pz}/2M_0 = 40.6/(2 \times 1750) = 0.0116 \text{ rad.}$$

Corresponding vertical deflection at hinge,

$$d_P = \phi L_{AD} L_{DB}/L_{AB} = 0.0116 \times 84 \times 36/120 = 0.29 \text{ in.}$$

If the structure were loaded so as to just reach the limit-design moment throughout the central sections of the girders, these would be bent in circular curves. In practice, even with perfectly symmetrical weight and structure, any component of longitudinal motion would lead to the formation of hinges at one end of the equipment. The assumption of a single hinge in each girder is therefore appropriate.

Longitudinal load criterion: For 12WF27 sections, with horizontal reactions at one end only, the maximum total of axial and bending stresses is

$$f_{max} = \frac{P_x}{2A} + \frac{P_x h L_{AC}}{2L_{AB} Z} = \frac{224}{2 \times 7.97} + \frac{224 \times 20 \times 36}{2 \times 120 \times 34.1}$$

$$= 14.1 + 19.7 = 33.8 < 45 \text{ k.s.i.}$$

Longitudinal energy criterion: This would not need to be investigated, since the scantlings are already determined and the specified longitudinal load does not give excessive stresses. The following calculations are to illustrate the method. The horizontal forces are assumed equally divided between the two ends of the equipment (with the girders supported horizontally at one end only). Shearing stresses are small and are neglected.

$$P_{Ex} = (45/33.8) \times 224 = 298\text{k.}$$

Axial stress in supported ends,

$$f_{AC} = P_{Ex}/2A = 298/(2 \times 7.97) = 18.7 \text{ k.s.i.}$$

Axial stress in central portions,

$$f_{CD} = f_{AC}/2 = 9.3 \text{ k.s.i.}$$

Bending stress at ends of equipment,

$$f_B = P_{Ex} h L_{AC}/2L_{AB} Z = 298 \times 20 \times 36/(2 \times 120 \times 34.1) = 26.2 \text{ k.s.i.}$$

From axial strains	2 × A in.^{2}	L in.	f k.s.i.	$(f/F_Y)^2$	$2AL(f/F_Y)^2$ in.^{3}
Supported ends	2 × 7.97	36	18.7	0.173	99
Central portions	2 × 7.97	48	9.3	0.043	33
Unsupported ends	2 × 7.97	36	0	0	0
From bending strains	**2 × Z/c in.^{2}**	**$\bar{L}$ in.**	**f_B k.s.i.**	**$(f_B/F_Y)^2$**	**$2(Z/c)\bar{L}(f_B/F_Y)^2$ in.^{3}**
Supported ends	2 × 5.70	36/3	26.2	0.339	46
Central portions	2 × 5.70	48/3	26.2	0.339	62
Unsupported ends	2 × 5.70	36/3	26.2	0.339	46
Structural parameter: $K_{Ex} = \Sigma AL(f/F_Y)^2 + \Sigma(Z/c)\bar{L}(f_B/F_Y)^2 = 286$ in.^{3}					

$$Q_{Ex} = K_{Ex}q = 286 \times 0.034 = 9.7 \text{ in.-k.}$$

Arrangement No. 4. – This is the same as Arrangement No. 3, except that the ends of the girders are attached to rigid structure. If the ends were truly fixed, the moment in way of the equipment, under vertical loading, would be only 43 percent of that at the ends and of opposite sign (see figure 6(b)). Under such conditions, the girders could absorb only a fraction of the energy that they can when simply supported. A limit-design condition, with end and field moments equal, is simpler and more realistic, provided the supporting structure is designed accordingly. With equal moments, bending energy is the same as for simple supports, while the design load is doubled. Permanent deformation, for the elastic-plastic design, is halved.

For the symmetrical arrangement of figure 5(d), with fixed ends,

$$K_{Ez} = 2\left(\frac{Z}{c}\right)_{AB}\left\{\left(\frac{2L_{AC}}{3}\right)\left[1 + \frac{M_C}{M_A} + \left(\frac{M_C}{M_A}\right)^2\right] + L_{CD}\left(\frac{M_C}{M_A}\right)^2\right\}$$

$$= (2 \times 5.70)\left\{(24)\left[1 - (0.43) + (0.43)^2\right] + (48)(0.43)^2\right\} = 308 < 820 \text{ in.}^3$$

A more realistic assumption is that the bending moment between load points is equal to that at the supports. Even if the supporting structure permitted no rotation of the girder ends, a redistribution of moments would take place before any significant plastic deformation occurs. Bending energy is the same as for Arrangement No. 3, but P_E is increased by a factor of 2. However, the webs require reinforcement to avoid excessive shear stresses.

Beyond the elastic range, three hinges would form prior to collapse of the structure. With the angle ϕ at one load point, angles at the supports would be $\phi(L_{DB}/L_{AB})$ and $\phi(L_{AD}/L_{AB})$. As a result, ϕ would be one-half the value found for Arrangement 3.

Arrangement No. 5. – Arrangement 5 is the same as Arrangement 3, except that allowance is made for vertical deformation of the supporting structure. Column supports are designed to take the simple-support reactions from the girder ends, under vertical loading. Their contribution to energy is found to be comparatively small.

The strength of the supports should be sufficient to develop the full plastic moment of the 12WF27 HTS girders. If a factor of safety of 1.25 is specified, each vertical leg must be able to withstand a load of at least

$$1.25\, M_0/L_{DB} = 1.25 \times 1750/36 = 60.8 \text{ k.}$$

For 2.875-inch O. D. x 0.203-inch thick HTS tubing,

$$A = 1.704 \text{ in.}^2,$$

$$L/r = 48/0.95 = 51,$$

$$F_C = 40 \text{ k.s.i.},$$

$$AF_C = 68.2 \text{ k.} > 60.8.$$

Axial stress in legs at limit-design load on beams:

$$f = M_0/L_{AC}A = 1750/(36 \times 1.704) = 28.5 \text{ k.s.i.}$$

Contribution of legs to K_{Ez}:

$$4AL(f/F_Y)^2 = 4 \times 1.704 \times 48(28.5/45)^2 = 130 \text{ in.}^3$$

Total (neglecting shear):

$$K_{Ez} = 820 + 130 = 950 \text{ in.}^3$$

The assumptions upon which these values are based are not strictly consistent, theoretically. The energy value for the beams corresponds to yield stress at the extreme fibers, rather than the limit design load. The discrepancy is insignificant from the point of view of practical design.

Arrangement No. 6. – Sometimes the shock resistance of a structure can be increased significantly by removing material in such a way that the distribution of stresses is more nearly uniform. Arrangement No. 6 illustrates the calculation of energy capacity for tapered girder ends.

Simpson's first rule is used to compute the contribution of each end section to K_{Ez}.

Station	M/M_{max}	Z in.^{3}	f_B/f_{max}	c in.	$\frac{Z}{c}\left(\frac{f_B}{f_{max}}\right)^2 \times$ Mult. in.^{2}
end	0	14.7	0	3.00	0 × 1
18 inches from end	0.5	24.1	.707	4.49	2.68 × 4
load point	1	34.1	1	5.98	5.70 × 1
Total	$\Sigma\left[\frac{Z}{c}\left(\frac{f_B}{f_{max}}\right)^2 \times \text{Mult.}\right] = 16.42$ in.^{2}				

$$\left(K_{Ez}\right)_{AC} = \frac{L_{AC}}{6}\,\Sigma\left[\frac{Z}{c}\left(\frac{f_B}{f_{max}}\right)^2 \times \text{Mult.}\right] = (36/6)(16.42) = 98.5 \text{ in.}^3$$

For the full-depth end section, the corresponding value is only

$$(L_{AC}/3)(Z/c) = (36/3)(5.70) = 68.4 \text{ in.}^3$$

9110-7-d. Analysis of Multi-Mass Systems for Specified Acceleration and Velocity Inputs.

If a foundation and the equipment it supports are to be considered as made up of several masses, a trial-and-error procedure must be used in lieu of the comparatively direct design method of 9110-7-c. This procedure consists of the following steps:

1. Preliminary design—that is, tentative selection of scantlings for the principal structural members.
2. The calculation of influence coefficients. The system must be divided into a suitable number of lumped masses. The deflection components of each are then computed for unit forces applied directly to it and, successively, to every other mass.
3. The determination of mode shapes and frequencies.
4. The calculation of shock forces resulting from the specified acceleration and velocity inputs.
5. The computation of stresses in the various structural members for comparison with allowable values. If stresses are excessive, scantlings must be changed and the whole process revised as necessary.

The above steps are described in detail below, and illustrated by examples.

Preliminary Design. – In general, scantlings will first be selected to suit an assumed load-weight ratio, applied uniformly to all masses.

If design inputs are given in terms of accelerations and velocities which decrease as modal weight($\bar{W}$) increases, it is apparent that the consideration of several modal weights will involve greater design values than would a single large mass having the same total weight. Also, because higher modes give alternating directions of loading, their combination with the first mode must be assumed to be the most adverse for each part of the structure. These factors should be allowed for when proportioning the structure. No general rule can be given for quantitatively evaluating the higher modes in advance of analysis, but the examples below may provide some clues.

Influence coefficients. – Deflections under unit loads can sometimes be calculated by handbook formulas. A more general procedure is as follows:

Bending deflections. – The deflection of point 1 due to a unit load at point 2 (or vice versa) can be found by integrating, over the entire structure, the value of $(m_1m_2/EI)dL$, where m_1 and m_2 are the bending moments due to unit loads 1 and 2 (in specific directions), respectively. The moments used in these calculations should, in general, be consistent with elastic structural response. However, assumptions which simplify analysis, such as those of the two-mass example, will frequently be justified.

If the curves of m_1, m_2, and I are smooth for a member AB whose midpoint is C, Simpson's first rule provides a convenient means of evaluating the integral. Then

$$\int_0^L \frac{m_1m_2}{EI}\,dL = \frac{L_{AB}}{6E}\left[\frac{m_{1A}m_{2A}}{I_A} + \frac{4m_{1C}m_{2C}}{I_C} + \frac{m_{1B}m_{2B}}{I_B}\right].$$

If m_1 and m_2 vary linearly, $(m_{1A} + m_{1B})/2$ may be substituted for m_{1C}, and $(m_{2A} + m_{2B})/2$ for m_{2C}. Then

$$\int_0^L \frac{m_1m_2}{EI}\,dL = \frac{L_{AB}}{6E}\left[\frac{m_{1A}m_{2A}}{I_A} + \frac{(m_{1A} + m_{1B})(m_{2A} + m_{2B})}{I_C} + \frac{m_{1B}m_{2B}}{I_B}\right].$$

Thus the only moments which need to be calculated for such a portion of beam are those at the ends.

If I is constant, the last formula reduces to

$$\int_0^L \frac{m_1m_2}{EI}\,dL = \frac{L_{AB}}{3EI_{AB}}\left[m_{1A}m_{2A} + \frac{m_{1A}m_{2B}}{2} + \frac{m_{1B}m_{2A}}{2} + m_{1B}m_{2B}\right].$$

To find the bending deflection of a load-point under a unit load (in the direction of the deflection) at that point, let $m_1 = m_2 = m$. Then for a uniform beam with straight-line moment curve

$$\int_0^L \frac{m^2}{EI}\,dL = \frac{L_{AB}}{3EI_{AB}}\left[m_A{}^2 + m_A m_B + m_B{}^2\right].$$

Deflections from axial stresses. – Similar expressions give deflections resulting from axial loads. That is, the deflection at point 1 from unit load at point 2 is given by the integral of $(p_1 p_2/AE)dL$, where p_1 and p_2 are the forces from unit loads 1 and 2, respectively. However, since axial forces change only at the mass locations and structural panel points, the effect of a portion of a uniform member, between load points, is $(L/AE)(p_1 p_2)$; $(L/AE)p^2$ for deflection at the load point.

Shear deflections. Unless the depth of a beam is small compared to the distances from supports to loads, shear deflections (and stresses) may be significant. If average stresses, based on web areas as in the approximate formula of 9110-7-c, are used, shear deflection 1 from unit load 2 can be obtained by evaluating, for each member,

$$\int_0^L \left(\frac{v_1 v_2}{A_w G}\right) dL,$$

where v_1 and v_2 are the shears due to unit loads 1 and 2, respectively. For each portion of beam of length L, of constant web area, and under constant shear, this integral reduces to

$$\frac{v_1 v_2 L}{A_w G}.$$

Total deflections. – Adding deflections resulting from strains, in all members, under the various kinds of stress, gives the influence coefficient, δ, for each combination of load and deflection. If a load in one direction causes significant deflections in other directions, two or three components may have to be considered. Each component gives the system a degree of freedom.

Mode shapes and natural frequencies. – If a structure is vibrating at a natural circular frequency, ω, the maximum inertia force acting on each mass is $(W/g)\omega^2 u$, where u is the amplitude of the mass in question. Under the action of such forces, the structure is deformed elastically. That is, for each mode of free vibration, the following equations must be satisfied.

$$u_1 = \frac{W_1}{g}\omega^2 u_1 \delta_{1-1} + \frac{W_2}{g}\omega^2 u_2 \delta_{1-2} + \dots + \frac{W_n}{g}\omega^2 u_n \delta_{1-n};$$

$$u_2 = \frac{W_1}{g}\omega^2 u_1 \delta_{2-1} + \frac{W_2}{g}\omega^2 u_2 \delta_{2-2} + \dots + \frac{W_n}{g}\omega^2 u_n \delta_{2-n};$$

$$\cdots\cdots\cdots$$

$$u_n = \frac{W_1}{g}\omega^2 u_1 \delta_{n-1} + \frac{W_2}{g}\omega^2 u_2 \delta_{n-2} + \dots + \frac{W_n}{g}\omega^2 u_n \delta_{n-n}.$$

For a two-degree-of-freedom system, the equations of motion are:

$$u_1 = \frac{W_1}{g}\omega^2 u_1 \delta_{1-1} + \frac{W_2}{g}\omega^2 u_2 \delta_{1-2};$$

$$u_2 = \frac{W_1}{g}\omega^2 u_1 \delta_{2-1} + \frac{W_2}{g}\omega^2 u_2 \delta_{2-2}.$$

These equations can be solved for two sets of compatible values of ω and the ratio, u_2/u_1.

$$\omega = \sqrt{\left(\frac{g}{W_1\delta_{1-1}}\right)\frac{\left(\frac{W_1}{W_2}+\frac{\delta_{2-2}}{\delta_{1-1}}\right) \pm \sqrt{\left(\frac{W_1}{W_2}+\frac{\delta_{2-2}}{\delta_{1-1}}\right)^2 - 4\left(\frac{W_1}{W_2}\right)\left[\frac{\delta_{2-2}}{\delta_{1-1}} - \left(\frac{\delta_{1-2}}{\delta_{1-1}}\right)^2\right]}}{2\left[\frac{\delta_{2-2}}{\delta_{1-1}} - \left(\frac{\delta_{1-2}}{\delta_{1-1}}\right)^2\right]}};$$

$$\frac{u_2}{u_1} = \left(\frac{W_1}{W_2}\right)\left(\frac{\delta_{1-1}}{\delta_{1-2}}\right)\left(\frac{g}{W_1\delta_{1-1}\omega^2} - 1\right).$$

Although these formulas appear rather complex, they can easily be solved from the three ratios, (W_1/W_2), $(\delta_{2-2}/\delta_{1-1})$, and $(\delta_{1-2}/\delta_{1-1})$, and the quantity, $(g/W_1\delta_{1-1})$, as in the two-mass example, below.

In general, algebraic solutions are impractical if the number of coupled modes is more than two. With the increasing availability of digital computers, iterative solutions are now widely used. One such procedure, using matrices, is suggested by NAVSHIPS 250-423-30. This work can be performed with desk calculators, although it becomes very tedious if many modes are involved.

The above equations can be written, for the first mode:

$$a_{1-1}u_{1-1} + a_{1-2}u_{2-1} + \cdots + a_{1-n}u_{n-1} = \left(g/\omega_1^2\right)\left(\Delta/\delta\right)u_{1-1}$$

$$a_{2-1}u_{1-1} + a_{2-2}u_{2-1} + \cdots + a_{2-n}u_{n-1} = \left(g/\omega_1^2\right)\left(\Delta/\delta\right)u_{2-1}$$

. . . .

$$a_{i-1}u_{1-1} + \cdots + a_{i-j}u_{i-1} + \cdots + a_{i-n}u_{n-1} = \left(g/\omega_1^2\right)\left(\Delta/\delta\right)u_{i-1}$$

. . . .

$$a_{n-1}u_{1-1} + a_{n-2}u_{2-1} + \cdots + a_{n-n}u_{n-1} = \left(g/\omega_1^2\right)\left(\Delta/\delta\right)u_{n-1}$$

where the general term, $a_{i-j} = W_j\Delta_{i-j}$, and Δ_{i-j} is the product of the influence coefficient, $\delta_{i-j}(=\delta_{j-i})$, and a constant. The latter is introduced for convenience in handling the very small influence coefficient values. The subscripts, i and j, denote row and column, respectively. Their maximum values are always equal (n). The subscript of ω and the second subscript of u indicate mode.

The "matrix" consists of the coefficients of u on the left sides of the equations. Assuming all u's to be unity (sometimes a more realistic mode shape can be used to advantage), the rows of the matrix are evaluated. If the true relative values of u had been known and used, the results would be proportional to the respective u values, and the constant, $D_1 = (g/\omega_1^2)(\Delta/\delta)$—and hence frequency—would be established. Of course the results of the first trial are likely to show a pattern very different from the assumed mode shape. The values are then "normalized" by dividing each by the largest value. This gives a new set of u's, for which the matrix is reevaluated. The process is repeated until results converge to show the true mode shape. For satisfactory checking, it is desirable to calculate mode shapes more precisely than would otherwise be justified. An example of a four-mass system, having five degrees of freedom, is given below.

Although the same equations are valid for higher modes, they must be transformed to obtain convergence. A column of coefficients is eliminated by utilizing the "orthogonality" relationship, which states that the summation of the products of the masses and the amplitudes for two different natural modes is equal to zero. For the first two modes this equation, when multiplied by g, becomes

$$W_1 u_{1-1} u_{1-2} + W_2 u_{2-1} u_{2-2} + \cdots + W_n u_{n-1} u_{n-2} = 0 .$$

One second-mode amplitude (relative deflection) can be expressed in terms of all the others and the known first-mode amplitudes.

$$u_{a-2} = \frac{W_1 u_{1-1}}{(-W_a u_{a-1})} u_{1-2} + \frac{W_2 u_{2-1}}{(-W_a u_{a-1})} u_{2-2} + \cdots + \frac{W_n u_{n-1}}{(-W_a u_{a-1})} u_{n-2} .$$

$W_a u_{a-1}$ is the largest (absolute) value of $W_j u_{j-1}$. Unavoidable discrepancies between various columns of the first-mode matrix will then not be magnified in the second-mode matrix.

The basic equations for the second mode are:

$$a_{1-1} u_{1-2} + a_{1-2} u_{2-2} + \cdots + a_{1-a} u_{a-2} + \cdots + a_{1-n} u_{n-2} = \left(g/\omega_2^2\right)\left(\Delta/\delta\right) u_{1-2}$$

$$a_{2-1} u_{1-2} + a_{2-2} u_{2-2} + \cdots + a_{2-a} u_{a-2} + \cdots + a_{2-n} u_{n-2} = \left(g/\omega_2^2\right)\left(\Delta/\delta\right) u_{2-2}$$

. . . .

$$a_{n-1} u_{1-2} + a_{n-2} u_{2-2} + \cdots + a_{n-a} u_{a-2} + \cdots + a_{n-n} u_{n-2} = \left(g/\omega_2^2\right)\left(\Delta/\delta\right) u_{n-2}$$

Substitution of the above value of u_{a-2} in these equations yields a second-mode matrix in which the coefficients of u_{a-2} are zero, and the other coefficients each contain an additional term. The general expression for the elements of this matrix may be written:

$$c_{i-j} = a_{i-j} + b_{i-j} ,$$

where

$$b_{i-j} = \frac{W_j u_{j-1}}{(-W_a u_{a-1})} a_{i-a} .$$

Relative values of the second-mode amplitudes and the constant, $D_2 \left[= \left(g/\omega_2^2\right)\left(\Delta/\delta\right)\right]$, can now be obtained by the same procedure as for the first mode.

Two orthogonality equations relate the third-mode amplitudes to those of the first and second.

$$\Sigma W_j u_{j-1} u_{j-3} = \Sigma W_j u_{j-2} u_{j-3} = 0 .$$

By eliminating u_{a-3}, these can be solved for some other third-mode amplitude.

$$u_{b-3} = \frac{W_1(u_{1-1}u_{a-2} - u_{a-1}u_{1-2})}{-W_b(u_{b-1}u_{a-2} - u_{a-1}u_{b-2})} u_{1-3} + \cdots + \frac{W_n(u_{n-1}u_{a-2} - u_{a-1}u_{n-2})}{-W_b(u_{b-1}u_{a-2} - u_{a-1}u_{b-2})} u_{n-3} .$$

This summation includes a term for each amplitude except a and b. Column b is chosen so that the common denominator is again not less than the numerators. By substitution, a third-mode matrix is obtained, in which two columns are zero. The general element is

$$e_{i-j} = c_{i-j} + d_{i-j},$$

where

$$d_{i-j} = \frac{W_j(u_{j-1}u_{a-2} - u_{a-1}u_{j-2})}{-W_b(u_{b-1}u_{a-2} - u_{a-1}u_{b-2})} c_{i-b}.$$

Similarly, a fourth-mode matrix can be obtained. u_{a-4} and u_{b-4} are eliminated from the three orthogonality equations, and the general term for the matrix is

$$g_{i-j} = e_{i-j} + f_{i-j},$$

where

$$f_{i-j} = \frac{W_j\left[(u_{b-2}u_{a-3} - u_{a-2}u_{b-3})u_{j-1} + (u_{a-1}u_{b-3} - u_{b-1}u_{a-3})u_{j-2} + (u_{b-1}u_{a-2} - u_{a-1}u_{b-2})u_{j-3}\right]}{-W_c\left[(u_{b-2}u_{a-3} - u_{a-2}u_{b-3})u_{c-1} + (u_{a-1}u_{b-3} - u_{b-1}u_{a-3})u_{c-2} + (u_{b-1}u_{a-2} - u_{a-1}u_{b-2})u_{c-3}\right]} e_{i-c}.$$

As before, the largest absolute value provides the basis for selection of column c.

For still higher modes, algebraic expressions for the solution of the simultaneous equations increase in complexity. The process of elimination and substitution can be carried out using only numerical coefficients, as illustrated in the fifth mode of the example. This does not, of course, shorten the process of obtaining new matrices. Fortunately, evaluations of the matrices become easier as columns and, for the trial calculations, rows are eliminated. For the nth mode of a system of n degrees of freedom, all relative amplitudes can be obtained directly from the orthogonality equations.

Effects of rotational inertia.—Rotational inertia may be significant, especially if the arrangement is unsymmetrical. One way of allowing for this is to divide the weight of the item in half, separating the two halves to give a corresponding moment of inertia about the center of gravity.

Shock forces.—The procedure followed here has been taken from NAVSHIPS 250-423-30 and NRL Memo Rept. 1396. Some formulas were modified to eliminate unnecessary computations.

To each mode being considered, an input acceleration is applied. This acceleration, which depends upon location, type of ship, and other factors, is assumed to be a function of "modal effective weight." The latter is defined by the formula,

$$\overline{W} = (\Sigma Wu)^2/\Sigma Wu^2.$$

In calculating the numerator, only deflections in the direction of the shock input are considered, but for the denominator, all deflections compatible with the given mode are included.

The smaller of two values, A or $V\omega/g$, but not less than a specified minimum, defines the input acceleration in g's. A and V are obtained from specified formulas or curves, and ω is the circular natural frequency of the mode in question.

The sum of the forces acting in the direction of the shock input is equal to the above acceleration times the modal weight. This quantity increases with increasing modal weight, although accelerations tend to decrease. The individual loads are calculated from the expression,

$$P = Wu(\Sigma Wu/\Sigma Wu^2)(A \text{ or } V\omega/g).$$

ΣWu, as in the formula for modal weight, includes only deflections in the direction of shock input, but all directions are included in the individual values of Wu and in the denominator, ΣWu^2. It should be noted that the calculations for mode shape and frequency indicate only relative deflections. All could be changed proportionally without affecting modal weight or the calculated forces. The sum of all modal weights, for loading in a given direction, is equal to the total of the weights considered free to move in that direction.

Design stresses.—It is assumed that the higher modes of vibration excited by the shock motion continue for a sufficient interval of time that stresses will be additive to those of the first mode. As the number of modes being considered is increased, however, the chance of coincidence of all maximum stresses is reduced. The assumptions to be made in combining stresses, as well as input motions and allowable stresses or factors of safety, will be stated in specifications.

Example of two-mass system.—Two masses, weighing 26 k. and 14 k. (including allowances for structure), are to be supported by a pair of unsymmetrical bents, of HTS, as shown in figure 7. Vertical shock inputs are to be taken from the following formulas:

$$A_z = \frac{16(37.5 + \bar{W})(12 + \bar{W})}{(6 + \bar{W})^2};$$

$$V_z = \frac{48(12 + \bar{W})}{(6 + \bar{W})}$$

For each mode, the smaller of the two values, A or $V\omega/g$, but not less than 6 is to be used. In the longitudinal direction (parallel to the beams), 20 percent of the above values will be used, with 6 again as the minimum. Athwartship forces are assumed to be resisted by other structure.

Preliminary design.—For a single mass weighing 40 k., the acceleration criteria give $A_z = 30.5$ g's and $A_x = 6.10$ g's.

The horizontal girder will be considered first on the basis of pin joints at A and B. Under vertical loading, the effective value of A_z, applied simultaneously to both masses, will exceed 30.5 g's. For $A_z = 40$, the reactions (for both bents) are:

$$R_E = (A_z)\,\frac{W_1 L_{CB} + W_2 L_{DB}}{L_{AB}} = (40)\,\frac{26 \times 106 + 14 \times 36}{130} = 1003 \text{ k.}$$

$$R_F = (A_z)\,\frac{W_1 L_{AC} + W_2 L_{AD}}{L_{AB}} = 597 \text{ k.}$$

$$\text{Check: } R_E + R_F = (A_z)(W_1 + W_2) = 1600 \text{ k.}$$

Bending moments (for both bents) are:

$$M_C = R_E L_{AC} = 1003 \times 24 = 24{,}100 \text{ in.-k.}$$

$$M_D = R_F L_{DB} = 597 \times 36 = 21{,}500 \text{ in.-k.}$$

Section modulus required to give yield stress at C is $Z_{AB} = M_C/2F_Y = 24{,}100/(2 \times 45) = 268$ in.^{3}, for each bent. 24WF110 gives $Z = 274$ in.^{3}.

If the legs are to be designed to give a factor of safety of 1.25 on column strength, the sectional area of AE must be at least $1.25\ R_E/2F_Y = 1.25 \times 1003/(2 \times 45) = 13.9$ in.^{2}, for each bent. 18WF50 gives $A = 14.7$ in.^{2}, $Z = 89$ in.^{3}, and $r = 1.59$ in. This section, used for both short and long legs, gives:

Member	AE	BF
L	35 in.	110 in.
L/r	22	69
F_C (DDS9110-4)	45 k.s.i.	35 k.s.i.
P	1003 k.	597 k.
P/2A	34.1 k.s.i.	20.3 k.s.i.
F.S.	1.31	1.72

Under the longitudinal loadings, if pin joints are assumed at A and B, with fixed ends at E and F, the full limit-design strength of the structure would be

$$P_0 = 2\left(\frac{Z_E}{L_{AE}} + \frac{Z_F}{L_{BF}}\right)F_Y.$$

However, sufficient movement to develop the full strength of BF would involve considerable yielding of AE. A safer assumption is that the short legs carry the entire load. The section modulus required to meet the acceleration criterion is then

$$Z_0 = A_x \frac{WL_{AE}}{2F_Y} = \frac{6.10 \times 40 \times 35}{(2 \times 45)} = 95 \text{ in.}^3$$

For the 18WF50, limit-design section modulus is about 14 percent greater than the elastic value, and $1.14 \times 89 = 101 > 95$ in.^{3}

It appears, then, that the scantlings shown on figure 7 will be adequate. The structure will now be analyzed as a two-mass system.

Influence coefficients

BENDING DEFLECTIONS FROM VERTICAL LOADS

Member		AC	CD	DB
Length, L – in.		24	70	36
3EI (Both Bents)– k.-in.^{2}		$3EI_{AB} = 3 \times 29{,}600 \times (2 \times 331.5) = 5.89 \times 10^8$		
(L/3EI) – (k.-in.)$^{-1}$		4.07×10^{-8}	11.88×10^{-8}	6.11×10^{-8}
Moments From Unit Loads – in.	Load 1 (at C)	$m_{1A} = 0$	$m_{1C} = 19.57$	$m_{1D} = 6.65$
		$m_{1C} = 19.57$	$m_{1D} = 6.65$	$m_{1B} = 0$
	Load 2 (at D)	$m_{2A} = 0$	$m_{2C} = 6.65$	$m_{2D} = 26.0$
		$m_{2C} = 6.65$	$m_{2D} = 26.0$	$m_{2B} = 0$

(Continued)

BENDING DEFLECTIONS FROM VERTICAL LOADS (Continued)

Member		AC	CD	DB
Influence Coefficients $\delta = \Sigma \frac{L}{3EI} \Sigma$ (Moment Products)	δ_{1-1}	$m_{1A}^2 = 0$	$m_{1C}^2 = 383$	$m_{1D}^2 = 44$
		$m_{1C}^2 = 383$	$m_{1D}^2 = 44$	$m_{1B}^2 = 0$
		$m_{1A}m_{1C} = 0$	$m_{1C}m_{1D} = 130$	$m_{1D}m_{1B} = 0$
		Σ Prod. = 383	Σ = 557	Σ = 44
	84.5 in./10^6 k.	$\times \frac{L}{3EI} = 15.6 \times 10^{-6}$	66.2×10^{-6}	2.7×10^{-6}
	δ_{2-2}	$m_{2A}^2 = 0$	$m_{2C}^2 = 44$	$m_{2D}^2 = 676$
		$m_{2C}^2 = 44$	$m_{2D}^2 = 676$	$m_{2B}^2 = 0$
		$m_{2A}m_{2C} = 0$	$m_{2C}m_{2D} = 173$	$m_{2D}m_{2B} = 0$
		Σ = 44	Σ = 893	Σ = 676
	149.2 in./10^6 k.	1.8×10^{-6}	106.1×10^{-6}	41.3×10^{-6}
	δ_{1-2}	$m_{1A}m_{2A} = 0$	$m_{1C}m_{2C} = 130$	$m_{1D}m_{2D} = 173$
		$m_{1C}m_{2C} = 130$	$m_{1D}m_{2D} = 173$	$m_{1B}m_{2B} = 0$
		$\frac{1}{2}m_{1A}m_{2C} = 0$	$\frac{1}{2}m_{1C}m_{2D} = 254$	$\frac{1}{2}m_{1D}m_{2B} = 0$
		$\frac{1}{2}m_{1C}m_{2A} = 0$	$\frac{1}{2}m_{1D}m_{2C} = 22$	$\frac{1}{2}m_{1B}m_{2D} = 0$
		Σ = 130	Σ = 579	Σ = 173
	84.7 in./10^6 k.	5.3×10^{-6}	68.8×10^{-6}	10.6×10^{-6}

SHEAR DEFLECTIONS FROM VERTICAL LOADS (APPROXIMATE)

Member		AC	CD	DB
Length, L – in.		24	70	36
GA_w (Both Bents) – k.		$11{,}400 \times (2 \times 11.45) = 2.61 \times 10^5$		
L/GA_w – in./k.		92.0×10^{-6}	268×10^{-6}	137.9×10^{-6}
Shears From Unit Loads	Load 1:	$v_1 = 0.815$	−0.185	−0.185
	Load 2:	$v_2 = 0.277$	0.277	−0.723
Influence Coefficients, $\delta = \Sigma \frac{L}{GA_w}$ (Shear Products)	δ_{1-1} 74.9 in./10^6 k.	$v_1^2 = 0.664$	0.034	0.034
		$\times \frac{L}{GA_w} = 61.1 \times 10^{-6}$	9.1×10^{-6}	4.7×10^{-6}
	δ_{2-2} 99.8 in./10^6 k.	$v_2^2 = 0.077$	0.077	0.523
		7.1×10^{-6}	20.6×10^{-6}	72.1×10^{-6}
	δ_{1-2} 25.6 in./10^6 k.	$v_1v_2 = 0.226$	−0.051	0.134
		20.8×10^{-6}	-13.7×10^{-6}	18.5×10^{-6}

AXIAL DEFLECTIONS FROM VERTICAL LOADS

Member		AE	BF
Length, L – in.		35	110
EA (Both Bents) – k.		$29{,}600 \times (2 \times 14.7) = 8.70 \times 10^5$	
L/EA – in./k.		40.2×10^{-6}	126.4×10^{-6}
Forces From Unit Loads	Load 1:	$P_1 = 0.815$	0.185
	Load 2:	$P_2 = 0.277$	0.723
Influence Coefficients, $\delta = \Sigma \frac{L}{EA}$ (Force Products)	δ_{1-1} 31.0 in./10^6 k.	$P_1^2 = 0.664$	0.034
		$\times \frac{L}{EA} = 26.7 \times 10^{-6}$	4.3×10^{-6}
	δ_{2-2} 69.2 in./10^6 k.	$P_2^2 = 0.077$	0.523
		3.1×10^{-6}	66.1×10^{-6}
	δ_{1-2} 26.0 in./10^6 k.	$P_1P_2 = 0.226$	0.134
		9.1×10^{-6}	16.9×10^{-6}

TWO-MASS SYSTEM; RESPONSE TO SHOCK INPUTS (VERTICAL)

Weights	$W_1 = 26.0$ k. (1)	$W_2 = 14.0$ k. (2)	
Influence Coefficients	$10^6\,\delta_{1-1} = 84.5 + 74.9 + 31.0 = 190$ (3)	$10^6\,\delta_{2-2} = 149.2 + 99.8 + 69.2 = 318$ (4)	$10^6\,\delta_{1-2} = 84.7 + 25.6 + 26.0 = 136$ (5)
Ratios	$\frac{W_1}{W_2} = 1.857$ (6)	$\frac{\delta_{2-2}}{\delta_{1-1}} = 1.674$ (7)	$\frac{\delta_{1-2}}{\delta_{1-1}} = 0.7158$ (8)
	(6) + (7) = 3.531 (9)	$(7) - (8)^2 = 1.162$ (10)	
	$\sqrt{(9)^2 - 4 \times (6) \times (10)} = 1.959$ (11)	$\frac{g}{W_1\delta_{1-1}} = \frac{386 \times 10^6}{(1) \times (3)} = 7.81 \times 10^4$ (12)	
	First Mode	Second Mode	
$\frac{\omega^2 W_1 \delta_{1-1}}{g}$	$\frac{(9) - (11)}{2 \times (10)} = 0.6764$ (13)	$\frac{(9) + (11)}{2 \times (10)} = 2.362$ (13A)	
Frequency	$\omega_1 = \sqrt{(12) \times (13)} = 230$ (14)	$\omega_2 = \sqrt{(12) \times (13A)} = 430$ (14A)	
Mode Shape $\frac{u_2}{u_1}$	$\frac{(6)}{(8)} \times \left(\frac{1}{(13)} - 1\right) = 1.241$ (15)	$\frac{(6)}{(8)} \times \left(\frac{1}{(13A)} - 1\right) = -1.496$ (15A)	

(Continued)

TWO-MASS SYSTEM; RESPONSE TO SHOCK INPUTS (VERTICAL) (Continued)

		First Mode				Second Mode			
	W	u	Wu	Wu^2	P = (19) × (Wu)	u	Wu	Wu^2	P = (19A) × (Wu)
W_1	26.0	1.000	26.00	26.00	725	1.000	26.00	26.00	237
W_2	14.0	(15) 1.241	17.37	21.56	485	(15A) −1.496	−20.94	31.33	−191
Σ	40.0	X	(16) 43.37	(17) 47.56	X	X	(16A) 5.06	(17A) 57.33	X
Modal Weights		$\bar{W}_1 = \frac{(16)^2}{(17)} = 39.55$				$\bar{W}_2 = \frac{(16A)^2}{(17A)} = 0.45$			Check: $\Sigma\bar{W} = 40.00$
Input Accel.	A	30.6 (18)				182			
Input Accel.	V; $\frac{V\omega}{g}$	54.3; 32.4				92.7; 103 (18A)			
$\frac{P}{Wu}$		$\frac{(16) \times (18)}{(17)} = 27.9$ (19)				$\frac{(16A) \times (18A)}{(17A)} = 9.1$ (19A)			

Calculation of stresses.—For the assumed structural arrangement, maximum values of the following are significant:

1. Forces at each mass (for design of attachments).
2. Reactions (for column stresses in legs and shears in beams).
3. Bending moments at C and D.

Reactions and moments for unit loads have already been determined. (See preceding tables.)

$P_1(\text{max.}) = 725 + 237 = 962$ k.

$P_2(\text{max.}) = 485 + 191 = 676$ k.

$R_E = 0.815\ P_1 + 0.277\ P_2$

$R_E(\text{max.}) = 0.815(725 + 237) + 0.277(485 - 191) = 865$ k.

$R_F = 0.185\ P_1 + 0.723\ P_2$

$R_F(\text{max.}) = 0.185(725 - 237) + 0.723(485 + 191) = 579$ k.

$M_C = 19.57\ P_1 + 6.65\ P_2$

$M_C(\text{max.}) = 19.57(725 + 237) + 6.65(485 - 191) = 20{,}800$ in.-k.

$M_D = 6.65\ P_1 + 26.0\ P_2$

$M_D(\text{max.}) = 6.65(725 - 237) + 26.0(485 + 191) = 20{,}800$ in.-k.

These reactions and moments are all somewhat less than the preliminary design values for vertical loading. However, maximum shear stress in the web of AB is found to exceed the allowable limit.

$$\frac{R_E}{2A_w} = \frac{865}{(2 \times 11.45)} = 37.8 > 0.6F_Y = 27 \text{ k.s.i.}$$

The addition of suitable reinforcement to the web of AB will have an effect on stiffness. If shear deflection is neglected, calculated frequencies are increased. For the first mode, this is not important since the controlling criterion was A_z, rather than V_z. However, the second mode is accentuated, and calculated reactions and moments are increased, as shown by the table below. The corresponding bending stress in AB exceeds yield, but only by a very slight margin.

If the beams were supported by equal legs having the same average length (72.5 in.) stresses would be reduced (see figure 8).

The table below shows, also, the possible effects of increasing the stiffness of the beams. If 15 square inches of deck plating is assumed to act in addition to the I section, I_{AB} is increased by 47 percent. Section modulus is also increased, but by only 11 percent. The result is an increase in second-mode response and somewhat higher stresses in the legs. Calculated stresses for the beams are somewhat lower, which would probably not have been the case if frequencies had not already been high enough that the acceleration criteria were dominant.

FORCES AND MOMENTS ON BENTS (VERTICAL SHOCK LOADING)

Quantities		Preliminary Design	Two-Mode Analyses Except as noted, legs are unequal, I-sections only are considered, and shear deflection is neglected.			
			Shear Included		Equal Legs	Plating Added
Influence Coefficients in./10^6 k.	$10^6\ \delta_{1-1}$		190	115	143	88.5
	$10^6\ \delta_{2-2}$		318	218	199	171
	$10^6\ \delta_{1-2}$		136	111	115	83.6
Frequencies rad./sec.	ω_1		230	274	265	313
	ω_2		430	654	618	717
Mode Shapes u_2/u_1	1st Mode		1.241	1.382	1.104	1.402
	2nd Mode		−1.496	−1.343	−1.682	−1.323
Modal Weights	W_1	40	39.55	39.00	39.91	38.90
	W_2		0.45	1.01	0.091	1.11
Forces on 26 k. Mass k.	1st Mode		±725	±689	±764	±684
	2nd Mode		±237	±551	±148	±616
	Max.	1,040	962	1,240	912	1,300
Forces on 14 k. Mass k.	1st Mode		±485	±513	±455	±516
	2nd Mode		∓191	∓399	∓134	∓439
	Max.	560	676	912	589	955

(Continued)

FORCES AND MOMENTS ON BENTS (VERTICAL SHOCK LOADING) (Continued)

Quantities		Preliminary Design	Two-Mode Analyses Except as noted, legs are unequal, I-sections only are considered, and shear deflection is neglected.			
			Shear Included		Equal Legs	Plating Added
Max. Reactions k.	R_E	1,003	865	1,043	832	1,081
	R_F	597	579	685	540	703
Max. Moments in.-k.	M_C	24,100	20,800	25,000	20,000	26,000
	M_D	21,500	20,800	24,600	19,400	25,300

Design as a rigid frame.—The foregoing calculations were based upon the assumption of pin joints at A and B. This resulted in a structure which, under vertical loadings, was statically determinate. With rigid joints, elastic analysis (by moment distribution) indicates the forces and moments shown by figure 9. The loads are increased slightly over those for the corresponding pin-jointed structure, but bending moments at C and D are reduced by the end moments.

While stresses in the beam are lowered, those in the legs are increased to calculated values in excess of yield strength. This does not mean that the structure would collapse. It implies some working of the joints, but actual strength is unlikely to be less than for a true pin-ended condition.

Under longitudinal loadings, rigid joints at A and B have the effect of doubling the limit-design load. For collapse of the structure, plastic hinges would have to form at A, B, E, and F. Figure 10 shows moment diagrams for the elastic and plastic conditions.

Even with pin-joints, the legs are sufficiently strong to meet the longitudinal acceleration criterion. And, since the specified A_x for this particular weight is close to the minimum value of 6, scantlings may not be materially reduced on the basis of the velocity criterion. As a matter of interest, however, discussion of this aspect of design is in order.

If the two masses are held together by structure which is stiff, compared to the legs, they may safely be treated as a single rigid body, as for the preliminary design. The energy corresponding to the specified velocity, in the longitudinal direction, is

$$Q_x = \frac{W(0.2V_z)^2}{2g} = \frac{40(10.85)^2}{2 \times 386} = 6.10 \text{ in.-k.}$$

The energy which can be absorbed elastically is calculated as follows. (See 9110-7-c.)

For AE, $L/6EI = 35/(6 \times 29{,}600 \times 2 \times 801) = 12.3 \times 10^{-8}$ (k.-in.)$^{-1}$

For AB, $L/6EI = 130/(6 \times 29{,}600 \times 2 \times 3315) = 11.0 \times 10^{-8}$ (k.-in.)$^{-1}$

For BF, $L/6EI = 110/(6 \times 29{,}600 \times 2 \times 801) = 38.7 \times 10^{-8}$ (k.-in.)$^{-1}$

From the rigid frame analysis, the load corresponding to yield stress at E gives the following bending moments (in in.-k., for both bents):

$$M_E = 8010, \quad M_A = -5280, \quad M_B = 1510, \quad M_F = -320$$

(The significance of the minus signs is that, for each member, the bending moment passes through 0.)

Calculation of Q_E:

$$\text{For AE, } 12.3 \times 10^{-8} (8010^2 - 8010 \times 5280 + 5280^2) = 6.12 \text{ in.-k.}$$
$$\text{For AB, } 11.0 \times 10^{-8} (5280^2 - 5280 \times 1510 + 1510^2) = 2.44 \text{ in.-k.}$$
$$\text{For BF, } 38.7 \times 10^{-8} (1510^2 - 1510 \times 320 \times 320^2) = \underline{0.74} \text{ in.-k.}$$
$$Q_{Ex} = 9.30 \text{ in.-k.}$$

With pin-joints at A and B and negligible axial deformation, the horizontal load would be distributed between the two pairs of legs in proportion to their stiffnesses. For a cantilever, $K = 3EI/L^3$.

$$\text{For AE, } K_A = 3 \times 29{,}600 \times 2 \times 801/35^3 = 3320 \text{ k./in.}$$
$$\text{For BF, } K_B = 3 \times 29{,}600 \times 2 \times 801/110^3 = 107 \text{ k./in.}$$

With yield stress at E, the moment at F would be

$$M_F = M_E \left(\frac{K_B}{K_A}\right)\left(\frac{L_{BF}}{L_{AE}}\right) = 8010 \left(\frac{107}{3320}\right)\left(\frac{110}{35}\right) = 811 \text{ in.-k.}$$

Then the energy is:

$$\text{For AE, } 12.3 \times 10^{-8} (8010)^2 = 7.89 \text{ in.-k.}$$
$$\text{For BF, } 38.7 \times 10^{-8} (811)^2 = \underline{0.25} \text{ in.-k.}$$
$$Q_{Ex} = 8.14 \text{ in.-k.}$$

Perhaps the most realistic assumption that could be made for this particular structure would be that $M_A = -M_E$ and that M_B and M_F are negligible. Since a true fixed-end could not exist at E, enough rotation to make $M_A = M_E$ seems justified. Then:

$$\text{For AE, } 12.3 \times 10^{-8} (8010)^2 = 7.89 \text{ in.-k.}$$
$$\text{For AB, } 11.0 \times 10^{-8} (8010)^2 = \underline{7.06} \text{ in.-k.}$$
$$Q_{Ex} = 14.95 \text{ in.-k.}$$

Conclusion.—The structure shown by figure 7, with rigid joints at A and B, is suitable to support the specified loads. Under vertical shock, the assumption of pin-joints at A and B is reasonable. For horizontal loading, a limit-design condition, with pins at B and F, gives the most realistic measure of response in terms of both strength and energy. While it might be possible to design a more uniformly stressed structure, this would probably be quite difficult, since moment patterns vary with direction of loading.

Example of multi-mass system.—The mode shapes, frequencies, and shock forces of a four-mass system, having five degrees of freedom, are calculated below. The arrangement, shown in figure 11, was chosen for simplicity in illustrating the procedure.

The following steps are numbered to correspond to the tabulated data.

Basic data.—

1. Number identifying deflection of lumped mass. Items are treated as separate masses only if the structure permits significant relative movement. The labor of calculation increases rapidly with the number of degrees of freedom.

2. Weights are rounded off to suit probable accuracy of data. In general, weights are not likely to be known closer than within one or two percent.

3. The influence coefficient matrix may be expressed in whatever units are convenient. In the example, only bending deflections are considered and $\Delta = (3EI \times 10^{-4})\delta$. Values of Δ are rounded off according to probable accuracy. This matrix is always symmetrical about its principal diagonal.

4. Columns of Δ's are totaled for checking purposes.

First mode.—

5. The first mode matrix is obtained by multiplying each column of Δ's by the corresponding weight. For best checking, exact products of the unavoidably approximate W's and Δ's are used.

6. Addition of the columns of a's should agree with multiplication of weights by the totals of step 4.

7. Rows of a's are totaled to give a first approximation of relative deflections. In effect, forces are based on all amplitudes being equal.

8. The values obtained in step 7 are divided by the sum which has the largest absolute value.

9. Successive approximations of the relative deflections are made. Rows of a's are multiplied, term by term, by the respective, normalized deflections just calculated. Their sums are then divided by the sum for the row selected to give the unit deflection. Hence it is convenient to calculate this row first. If later approximations show some other amplitude to be greater, the basis for normalization may be changed at any time. However this is not essential. As the values converge, significant figures are added.

10. At this point it is advisable to verify the influence coefficients by computing the first-mode forces, bending moments, and deflections. Force is proportional to weight and amplitude.

Second mode.—

11. Each weight is multiplied by the corresponding final value of step 9. The column for which this product has the largest absolute value is designated (a).

12. Each product, from 11, is divided by the negative of 11(a). Hence 12(a) = −1.

13. Column (a) of step 5 is multiplied, successively, by the values found in step 12. Column (a) of 13 is, of course, numerically equal to column (a) of 5, but note that the signs are reversed.

14. The columns of 13 are added and checked by multiplying Σa_{i-a} (from 6) by the values of 12.

15. The elements of 13 are added, algebraically, to the corresponding elements of 5. In the resulting second-mode matrix, one column (for j = a) is reduced to zero.

16. The addition and checking are similar to step 6.

17. Rows are added, as in 7.

18. Results are normalized, as in 8. However, trial values of u_{a-2} are of no significance since their new coefficients are zero. To save computations, normalization is based on deflection at another point, regardless of the value of u_{a-2}.

19. Successive approximations of the relative amplitudes are made, as in step 9, until the results converge. u_{a-2} is omitted until the final step.

20. As a check, the final values of 19 are substituted in the matrix of 5. The divisor for normalization and the relative amplitudes should agree, approximately, with step 19. However, slight inaccuracies in the mode shape will produce relatively large apparent discrepancies in the first-mode matrix. The final results of step 19 are used as the values of u_{j-2} in the subsequent calculations.

Third mode.—

21. For each column, $W_j(u_{a-2}u_{j-1} - u_{a-1}u_{j-2})$ is calculated. It is zero when j = a. The column corresponding to the largest absolute value is designated (b).

22 through 30. These steps are similar to the corresponding ones for the second mode, 12 through 20. In the third-mode matrix (25) two columns, for j = a and j = b, are zero. Amplitudes u_{a-3} and u_{b-3} are omitted until the final trial of step 29.

Fourth mode.—

31. The expression to be evaluated, for each column, is

$$W_j\left[(u_{b-2}u_{a-3} - u_{a-2}u_{b-3})u_{j-1} + (u_{a-1}u_{b-3} - u_{b-1}u_{a-3})u_{j-2} + (u_{b-1}u_{a-2} - u_{a-1}u_{b-2})u_{j-3}\right].$$

It is zero for both (a) and (b), and the largest absolute value provides the basis for selecting (c).

32 through 40. These steps follow the corresponding ones for the lower modes. Three columns have now been eliminated from the matrix, and three rows are evaluated only to give the final values of u_{a-4}, u_{b-4}, and u_{c-4}.

Fifth mode.—

41. The coefficients of u_{j-5} are here determined, numerically, by eliminating the terms involving u_{a-5}, u_{b-5}, and u_{c-5}. Since only two coefficients remain, the final mode shape is readily apparent. Amplitudes are inversely proportional to the coefficients.

42 through 47. To illustrate the procedure more generally, the data from step 41 are treated in the same manner as for the lower modes.

48. Since only one column remains in the fifth-mode matrix, normalization of the values found in 47 defines the mode shape.

49 An alternative procedure for the final mode is to solve the equations of step 41 for all relative values.

50. Values of u_{j-5} are substituted in the first-mode matrix. The results from step 49 give better agreement than steps 42 through 48.

Response to shock inputs.—For vertical shock motion, forces and the resulting reactions and bending moments are calculated below, for all five modes. Effects are combined by adding the greatest value and the square root of the sum of the squares of all the others.

For horizontal shock motion, the numerator of the expression for modal weight involves only the values for $j = 5$, which are listed in parentheses. Weights calculated are: 1.361, 3.225, 0.455, 0.139, and 1.020, giving the correct total of 6.200. If A and V are to be reduced to 20 percent of their values for vertical shock, the five ratios of P/Wu are: 2.3, 10.7, 9.2, 5.6, and 11.8. These are all less than the corresponding ratios for the vertical inputs. It is therefore unnecessary to calculate forces, reactions, and moments.

Accelerating convergence.—After the first few iterations, increments of the calculated values of a relative displacement tend to decrease by a constant ratio. For example, successive values of u_{3-3} (Step 29) are 0.6925, 0.6833, 0.6784, 0.67572 . . . Their differences are 0.0092, 0.0049, 0.00268 . . . ; and the ratios of successive differences are 0.53, 0.55 . . .

If it is assumed that this ratio is a constant, numerically less than 1, an extrapolation can be made for the limiting value of the displacement. The result is a short cut in the iteration process which may be well worth taking when determining mode shapes with a desk calculator. If three successive values of displacement are u_0, u_1, and u_2,

$$u \to u_1 + \frac{(u_2 - u_1)}{1 - \frac{(u_2 - u_1)}{(u_1 - u_0)}} .$$

From the first three of the values listed above,

$$u_{3-3} = 0.6833 + \frac{0.6784 - 0.6833}{1 - \frac{0.6784 - 0.6833}{0.6833 - 0.6925}} = 0.6728 .$$

The final value of u_{3-3} (0.6723 . . .) differs by only 0.0005, less than a twelfth of its difference from 0.6784. Four more iterations were required to obtain a comparable value without the short cut.

The corresponding three values of u_{4-3}, −0.0224, −0.0239, and −0.0241, give

$$u_{4-3} = -0.0239 + \frac{(-0.0241) - (-0.0239)}{1 - \frac{(-0.0241) - (-0.0239)}{(-0.0239) - (-0.0224)}} = 0.0241 .$$

This result is somewhat misleading as the sign of the increment of u_{4-3} reverses on the very next iteration. However, the discrepancy is automatically corrected in the subsequent operations. The tabulated data, step 29A, show that two extrapolations saved seven iterations.

CALCULATION OF MODE SHAPES
5 DEGREES OF FREEDOM

Basic Data

1	j (Column No.)		1	2	3	4 (Vert.)	5 (Horiz.)
2	W_j – k.		3.1	13.6	7.5	6.2	
3	Δ_{i-j} $\left(\frac{\Delta}{\delta} = \frac{3EI}{10^4}\right)$	i=1	24.0	–21.5	–15.7	8.3	–13.0
		2	–21.5	33.1	26.6	–14.3	22.2
		3	–15.7	26.6	28.8	–16.8	26.2
		4	8.3	–14.3	–16.8	15.6	–25.8
		5	–13.0	22.2	26.2	–25.8	47.3
4	$\Sigma\Delta$		–17.9	46.1	49.1	–33.0	56.9

First Mode

5	$a_{i-j} = W_j\Delta_{i-j}$	i=1	74.400	–292.400	–117.750	51.460	–80.600
		2	–66.650	450.160	199.500	–88.660	137.640
		3	–48.670	361.760	216.000	–104.160	162.440
		4	25.730	–194.480	–126.000	96.720	–159.960
		5	–40.300	301.920	196.500	–159.960	293.260
6	Σa (Columns)		–55.490	626.960	368.250	–204.600	352.780
7	Σa (Rows)		–364.890	631.990	587.370	–357.990	591.420
8	Normalization		–0.577	1	0.929	–0.566	0.936
9	D_i; u_{j-1} — Divisor	853.0	–0.644	1	0.940	–0.622	1.023
		912.1	–0.620	0.961	0.907	–0.607	1
		883.7	–0.6175	0.9577	0.9046	–0.6063	1
		882.07	–0.61706	0.95700	0.90412	–0.60621	1
		881.69	–0.61699	0.95690	0.90407	–0.60623	1
		881.654	–0.616968	0.956873	0.904052	–0.606225	1
		881.641	–0.616964	0.956867	0.904048	–0.606224	1
		881.638	–0.616963	0.956866	0.904047	–0.606224	1

10	Location	A	B	C	D	E	F (Vert.)	G (Horiz.)
	Relative Force	–1.913	–10.274	13.013	6.780	–3.847	–3.759	6.200
	Moment	0	–63.1	–843.1	–787.8	–361.9	–260.4	0
	Relative Defl.	–0.6176	0	0.9578	0.9043	0	–0.6064	1
	Difference	0.0006	–	0.0009	0.0003	–	0.0002	–

Second Mode

1	j		1	2(a)	3	4	5
11	$W_j u_{j-1}$		-1.91259	13.01338	6.78035	-3.75859	6.2
12	$W_j u_{j-1}/(-W_a u_{a-1})$		0.146971	-1	-0.521029	0.288825	-0.476433
13	b_{i-j} = (12) × a_{i-a}	i=1	-42.974	292.400	152.349	-84.452	139.309
		2	66.160	-450.160	-234.546	130.017	-214.471
		3	53.168	-361.760	-188.487	104.485	-172.354
		4	-28.583	194.480	101.330	-56.171	92.657
		5	44.373	-301.920	-157.309	87.202	-143.845
14	Σb (Columns)		92.144	-626.960	-326.663	181.081	-298.704
15	$c_{i-j} = a_{i-j} + b_{i-j}$	i = 1	31.426	0	34.599	-32.992	58.709
		2	-0.490	0	-35.046	41.357	-76.831
		3	4.498	0	27.513	0.325	-9.914
		4	-2.853	0	-24.670	40.549	-67.303
		5	4.073	0	39.191	-72.758	149.415
16	Σc (Columns)		36.654	0	41.587	-23.519	54.076
17	Σc (Rows)		91.742	-71.010	22.422	-54.277	119.921
18	Normalization		0.765	–	0.187	-0.452	1
19		Divisor 192.75	0.540	–	-0.008	-0.480	1
		186.22	0.490	–	-0.042	-0.473	1
		184.18	0.4792	–	-0.0490	-0.4715	1
		183.75	0.4769	–	-0.0504	-0.4712	1
		183.666	0.47640	–	-0.05068	-0.47111	1
		183.6462	0.476295	–	-0.050742	-0.471096	1
		183.6423	0.476273	–	-0.050755	-0.471093	1
		183.6415	0.476268	–	-0.050758	-0.471092	1
	D_2; u_{j-2}	183.6413	0.476267	-0.516052	-0.050758	-0.471092	1
20	$\Sigma a_{i-j} u_{j-2}$	183.6419	0.476265	-0.516050	-0.050760	-0.471093	1
	Difference	0.0006	0.000002	0.000002	0.000002	0.000001	–

Third Mode

1	j		1	2(a)	3	4	5(b)
21	$u_{a-2} u_{j-1} = (-0.516052) u_{j-1}$		0.318385	-0.493793	-0.466535	0.312843	-0.516052
	$-u_{a-1} u_{j-2} = (-0.956866) u_{j-2}$		-0.455724	0.493793	0.048569	0.450772	-0.956866
	$W_j(u_{a-2} u_{j-1} - u_{a-1} u_{j-2})$		-0.425751	0	-3.134745	4.734413	-9.132092
22	$(21)_j / -(21)_b$		-0.046621	0	-0.343267	0.518437	-1
23	d_{i-j} = (22) × c_{i-b}	i=1	-2.737	0	-20.153	30.437	-58.709
		2	3.582	0	26.374	-39.832	76.831
		3	0.462	0	3.403	-5.140	9.914
		4	3.138	0	23.103	-34.892	67.303
		5	-6.966	0	-51.289	77.462	-149.415

(Continued)

1	j		1	2(a)	3		
24	Σd (Columns)		−2.521	0	−18.562	28.035	
25	$e_{i-j} = c_{i-j} + d_{i-j}$	i 1	28.689	0	14.446	−2.555	
		2	3.092	0	−8.672	1.525	0
		3	4.960	0	30.916	−4.815	0
		4	0.285	0	−1.567	5.657	0
		5	−2.893	0	−12.098	4.704	0
26	Σe (Columns)		34.133	0	23.025	4.516	0
27	Σe (Rows)		40.580	−4.055	31.061	4.375	−10.287
28	Normalization		1	−	0.765	0.108	−
29	Divisor	39.464	1	−	0.7118	−0.0077	−
		38.991	1	−	0.6925	−0.0224	−
		38.750	1	−	0.6833	−0.0239	−
		38.621	1	−	0.6784	−0.0241	−
		38.5507	1	−	0.67572	−0.02372	−
		38.5111	1	−	0.67422	−0.02358	−
		38.4890	1	−	0.67338	−0.02351	−
		38.4767	1	−	0.67291	−0.02347	−
		38.4698	1	−	0.67265	−0.02345	−
		38.4660	1	−	0.67250	−0.02344	−
		38.46382	1	−	0.672421	−0.023435	−
		38.46267	1	−	0.672377	−0.023432	−
		38.46203	1	−	0.672352	−0.023430	−
		38.46166	1	−	0.672338	−0.023429	−
		38.46146	1	−	0.672330	−0.023428	−
		38.46134	1	−	0.672326	−0.023428	−
		38.46128	1	−	0.672324	−0.023428	−
	$D_3; u_{j-3}$	38.46125	1	−0.072128	0.672323	−0.023428	−0.289564
30	$\Sigma a_{i-j} u_{j-3}$	38.4574	1	−0.07201	0.67249	−0.02347	−0.28955
	Difference	0.0039	−	0.00012	0.00017	0.00004	0.00001

Fourth Mode

1	j		1	2(a)	3(c)	4	5(b)
31	$(u_{b-2}u_{a-3} - u_{a-2}u_{b-3})u_{j-1}$		0.136693	−0.212001	−0.200299	0.134314	−0.221558
	$(u_{a-1}u_{b-3} - u_{b-1}u_{a-3})u_{j-2}$		−0.097609	0.105763	0.010403	0.096548	−0.204946
	$(u_{b-1}u_{a-2} - u_{a-1}u_{b-2})u_{j-3}$		−1.472918	0.106239	−0.990277	0.034508	0.426504
	W_j × Sum		−4.44489	0	−8.85130	1.64529	0
32	(31)$_j$/−(31)$_c$		−0.50217	0	−1	0.18588	0
33	f_{i-j} = (32) × e_{i-c}	i=1	−7.254	0	−14.446	2.685	0
		2	4.355	0	8.672	−1.612	0
		3	−15.525	0	−30.916	5.747	0
		4	0.787	0	1.567	−0.291	0
		5	6.075	0	12.098	−2.249	0

(Continued)

Fourth Mode (Continued)

l	j		1	2(a)	3(c)	4	5(b)
34	Σf (Columns)		−11.562	0	−23.025	4.280	0
35	$g_{i-j} = e_{i-j} + f_{i-j}$	i = 1	21.435	0	0	0.130	0
		2	7.447	0	0	−0.087	0
		3	−10.565	0	0	0.932	0
		4	1.072	0	0	5.366	0
		5	3.182	0	0	2.455	0
36	Σg (Columns)		22.571	0	0	8.796	0
37	Σg (Rows)		21.565	7.360	−9.633	6.438	5.637
38	Normalization		1	–	–	0.299	–
39		Divisor 21.474	1	–	–	0.125	–
		21.451	1	–	–	0.081	–
		21.446	1	–	–	0.07025	–
		21.444	1	–	–	0.06757	–
		21.444	1	–	–	0.06690	–
		21.4437	1	–	–	0.06673	–
		21.4437	1	–	–	0.066690	–
		21.4437	1	–	–	0.066680	–
		21.4437	1	–	–	0.066677	–
	D_4; u_{j-4}	21.4437	1	0.347011	−0.489788	0.066676	0.156022
40	$\Sigma a_{i-j} u_{j-4}$	21.4623	1	0.34531	−0.49064	0.06748	0.15449
	Difference	0.0186	–	0.00170	0.00085	0.00080	0.00153

Fifth Mode

l	j		1(d)	2(a)	3(c)	4	5(b)
41	a	$W_j u_{j-1}$	−1.91259	13.01338	6.78035	−3.75859	6.2
	b	$W_j u_{j-2}$	1.47643	−7.01831	−0.38069	−2.92077	6.2
	c	$W_j u_{j-3}$	3.1	−0.98094	5.04242	−0.14525	−1.79530
	d	$W_j u_{j-4}$	3.1	4.71935	−3.67341	0.41339	0.96734
	e	From (a)	−0.14417	0.98094	0.51110	−0.28332	0.46735
	f	From (b)	−0.20636	0.98094	0.05321	0.40823	−0.86657
	g	From (d)	0.64435	0.98094	−0.76354	0.08593	0.20107
	h	(c) + (e)	2.95583	0	5.55352	−0.42857	−1.32795
	i	(c) + (f)	2.89364	0	5.09563	0.26298	−2.66187
	j	(c) + (g)	3.74435	0	4.27888	−0.05932	−1.59423
	k	From (i)	−1.44358	0	−2.54210	−0.13120	1.32795
	l	From (j)	−3.11894	0	−3.56419	0.04941	1.32795
	m	(h) + (k)	1.51225	0	3.01142	−0.55977	0
	n	(h) + (l)	−0.16311	0	1.98933	−0.37916	0
	o	From (m)	−0.99899	0	−1.98933	0.36978	0
	p	(n) + (o)	−1.16210	0	0	−0.00938	0

(Continued)

Fifth Mode (Continued)

1	j		1(d)	2(a)	3(c)	4	5(b)
42	(41)$_j$/−(41)$_d$		−1	0	0	−0.00807	0
43	h_{i-j} = (42) × g_{i-d}	i=1	−21.435	0	0	−0.173	0
		2	−7.447	0	0	−0.060	0
		3	10.565	0	0	0.085	0
		4	−1.072	0	0	−0.009	0
		5	−3.182	0	0	−0.026	0
44	Σh (Columns)		−22.571	0	0	−0.183	0
45	$i_{i-j} = g_{i-j} + h_{i-j}$	1	0	0	0	−0.043	0
		2	0	0	0	−0.147	0
		3	0	0	0	1.017	0
		4	0	0	0	5.357	0
		5	0	0	0	2.429	0
46	Σi (Columns)		0	0	0	8.613	0
47	Σi (Rows)		−0.043	−0.147	1.017	5.357	2.429
48	u_{j-5}; D_5 = 5.357		−0.00803	−0.02744	0.18985	1	0.45343
49	u_{j-5}	From Equation	(p)	(a)	(m)	(p)	(i)
			−0.00807	−0.02743	0.18993	1	0.45361
50	Σa_{i-j} (49)	D_5 = 5.356	−0.00842	−0.02691	0.19025	1	0.45378
	Difference	0.001	0.00035	0.00052	0.00032	–	0.00017

Response to Vertical Shock Inputs

Formulas and Criteria

Frequency: $\omega = \sqrt{\dfrac{g\Delta/\delta}{D}}$

$g\Delta/\delta = 386 \times 3EI/10^4 = 46.1 \times 10^6$

Modal Weight: $\overline{W} = (\Sigma Wu^*)^2/\Sigma Wu^2$

Acceleration: $A = 16\,\dfrac{(37.5 + \overline{W})(12 + \overline{W})}{(6 + \overline{W})^2}$

Velocity: $V = 48\,\dfrac{(12 + \overline{W})}{(6 + \overline{W})}$

Load: $P = Wu(\pm\Sigma Wu^*/\Sigma Wu^2) \times$ (A or $V\omega/g$, whichever is less)

*Deflections in direction of shock loading only.

First Mode

$\omega_1 = \sqrt{46.1 \times 10^6/881.6} = 229$

j	W_j	u_{j-1}	$W_j u_{j-1}$	$W_j u_{j-1}^2$	P_{j-1}
1	3.1	−0.6170	−1.913	1.180	−40
2	13.6	0.9569	13.014	12.453	271
3	7.5	0.9040	6.780	6.129	141
4	6.2	−0.6062	−3.758	2.278	−78
5	(6.2)	1	(6.2)	6.200	129
Σ	30.4		14.123	28.240	

$\overline{W}_1 = (14.123)^2/28.40 = 7.063$ k.

$A_1 = 79.7$ $\quad V_1\omega_1/g = 41.6$

$P_{j-1}/W_j u_{j-1} = 20.8$

Response to Vertical Shock Inputs (Continued)

	Second Mode				Third Mode			
	$\omega_2 = \sqrt{46.1 \times 10^6/183.6} = 501$				$\omega_3 = \sqrt{46.1 \times 10^6/38.46} = 1095$			
j	u_{j-2}	$W_j u_{j-2}$	$W_j u^2_{j-2}$	P_{j-2}	u_{j-3}	$W_j u_{j-3}$	$W_j u^2_{j-3}$	P_{j-3}
1	0.4763	1.477	0.703	−91	1	3.100	3.100	247
2	−0.5161	−7.019	3.623	−432	−0.0721	−0.981	0.071	−78
3	−0.0508	−0.381	0.019	23	0.6723	5.042	3.390	401
4	−0.4711	−2.921	1.376	180	−0.0234	−0.145	0.003	−12
5	1	(6.2)	6.200	−381	−0.2896	(−1.796)	0.520	−143
Σ		−8.844	11.921			7.016	7.084	

$\bar{W}_2 = 6.561$ k.
$A_2 = 82.9 \quad V_2\omega_2/g = 92$
$P_{j-2}/W_j u_{j-2} = 61.5$

$\bar{W}_3 = 6.949$ k.
$A_3 = 80.4 \quad V_3\omega_3/g = 199$
$P_{j-3}/W_j u_{j-3} = 79.6$

	Fourth Mode				Fifth Mode			
	$\omega_4 = \sqrt{46.1 \times 10^6/21.46} = 1466$				$\omega_5 = \sqrt{46.1 \times 10^6/5.36} = 2930$			
j	u_{j-4}	$W_j u_{j-4}$	$W_j u^2_{j-4}$	P_{j-4}	u_{j-5}	$W_j u_{j-5}$	$W_j u^2_{j-5}$	P_{j-5}
1	1	3.100	3.100	249	−0.0081	−0.025	0.000	−2
2	0.3470	4.719	1.637	379	−0.0274	−0.373	0.010	−28
3	−0.4898	−3.674	1.800	−295	0.1899	1.424	0.270	109
4	0.0667	0.414	0.028	33	1	6.200	6.200	472
5	0.1560	(0.967)	0.151	78	0.4536	(2.812)	1.276	214
Σ		4.559	6.716			7.226	7.756	

$\bar{W}_4 = 3.095$ k.
$A_4 = 118.4 \quad V_4\omega_4/g = 302$
$P_{j-4}/W_j u_{j-4} = 80.4$

$\bar{W}_5 = 6.732$ k.
$A_5 = 81.8 \quad V_5\omega_5/g = 536$
$P_{j-5}/W_j u_{j-5} = 76.2$

Effects of Loads; Vertical Shock

	Load Pts	Bending Moments					Reactions	
		B	C	D	E	F	B	E
Unit Loads	1	33.00	21.71	9.88	0	0	1.1765	−0.1765
	2	0	−42.10	−19.17	0	0	0.6578	0.3422
	3	0	−19.17	−39.23	0	0	0.2995	0.7005
	4	0	9.24	18.91	27.00	0	−0.1444	1.1444
	5	0	−14.37	−29.42	−42.00	−42.00	−0.2246	0.2246

(Continued)

Effects of Loads; Vertical Shock (Continued)

Mode	Load Pts	Bending Moments B	C	D	E	F	Reactions B	E
1	1	−1,320	−870	−400	0	0	−47	7
	2	0	−11,410	−5,200	0	0	178	93
	3	0	−2,700	−5,530	0	0	42	99
	4	0	−720	−1,470	−2,110	0	11	−89
	5	0	−1,850	−3,800	−5,420	−5,420	−29	29
	Σ	∓1,320	∓17,550	∓16,400	∓7,530	∓5,420	±155	±139
2	1	−3,000	−1,980	−900	0	0	−107	16
	2	0	−18,190	−8,280	0	0	284	148
	3	0	−440	−900	0	0	7	16
	4	0	1,660	3,400	4,860	0	−26	206
	5	0	5,470	11,210	16,000	16,000	86	−86
	Σ	∓3,000	∓13,480	±4,530	±20,860	±16,000	±244	±300
3	1	8,150	5,360	2,440	0	0	291	−44
	2	0	3,280	1,500	0	0	−51	−27
	3	0	−7,690	−15,730	0	0	120	281
	4	0	−110	−230	−320	0	2	−14
	5	0	2,050	4,210	6,010	6,010	32	−32
	Σ	±8,150	±2,890	∓7,810	±5,690	±6,010	±394	±164
4	1	8,220	5,410	2,460	0	0	293	−44
	2	0	−15,960	−7,270	0	0	249	130
	3	0	5,660	11,570	0	0	−88	−207
	4	0	300	620	890	0	−5	38
	5	0	−1,120	−2,290	−3,280	−3,280	−18	18
	Σ	±8,220	∓5,710	±5,090	∓2,390	∓3,280	±431	∓65
5	1	−70	−40	−20	0	0	−2	0
	2	0	1,180	540	0	0	−18	−10
	3	0	−2,090	−4,280	0	0	33	76
	4	0	4,360	8,930	12,740	0	−68	540
	5	0	−3,080	−6,300	−8,990	−8,990	−48	48
	Σ	∓70	±330	∓1,130	±3,750	∓8,990	∓103	±654

Summary of Responses

Mode		1	2	3	4	5	Max.	$\sqrt{\Sigma Sq.}$	Total
Forces	1	40	91	247	249	2	249	266	515
	2	271	432	78	379	28	432	473	905
	3	141	23	401	295	109	401	345	746
	4	78	180	12	33	472	472	199	671
	5	129	381	143	78	214	381	298	679

(Continued)

Summary of Responses (Continued)

Mode		1	2	3	4	5	Max.	$\sqrt{\Sigma Sq.}$	Total
Moments	B	1,320	3,000	8,150	8,220	70	8,220	8,780	17,000
	C	17,550	13,480	2,890	5,710	330	17,550	15,280	32,830
	D	16,400	4,530	7,810	5,090	1,130	16,400	12,130	28,530
	E	7,530	20,860	5,690	2,390	3,750	20,860	13,250	34,110
	F	5,420	16,000	6,010	3,280	8,990	16,000	12,530	28,530
Reactions	B	155	244	394	431	103	431	499	930
	E	139	300	164	65	654	654	375	1,029

Third Mode; Accelerated Convergence

1	j		1	2	3	4	5
	Extrapolation		1	–	0.6728	–0.0241	–
	Divisor	38.4698	1	–	0.672640	–0.023541	–
		38.4661	1	–	0.672506	–0.023454	–
		38.4639	1	–	0.672426	–0.023437	–
29A	Extrapolation		1	–	0.672307	–0.023433	–
	Divisor	38.4610	1	–	0.672314	–0.023428	–
		38.4611	1	–	0.672317	–0.023428	–
		38.46115	1	–	0.672319	–0.023428	–
		38.46118	1	–0.072127	0.672320	–0.023428	–0.289563

9110-7-e. Shock Design Spectra

Significance of spectra.–Shock requirements can be described by a spectrum, where limiting design values are defined as functions of natural frequency. The criteria used in the foregoing articles imply design spectra made up of straight lines, as in figure 12.

Any point on figure 12 represents a set of related properties of a single-degree-of-freedom system:

Natural frequency, n.

Energy-weight ratio, Q/W (= $V^2/2g$. Ordinates can also be read in terms of velocity, directly.)

Load-weight ratio, P/W (= acceleration in g's, A)

Deflection, d (movement of center of gravity in direction of loading)

If any two of the above are known, the other two can be determined, either graphically or by the formulas already stated in 9110-7-c.

The required value of Q/W, based on a step velocity change, is represented by a horizontal line across the spectrum. If the structure is stiff enough, however, the diagonal corresponding to the limiting acceleration, P/W, shows that a lesser volume of material will suffice.

Under a typical, actual shock excitation, strains may be amplified if the frequency of the equipment on its foundation is close to resonance with a natural frequency of the hull or of the bulkhead or deck to which the foundation is attached. Hence there are peaks in the spectrum at these frequencies. Since the shock excitation cannot be accurately defined before the ship is built, and, in general, foundation frequencies and responses (particularly those beyond the elastic limit) can only be approximated, design must usually be based on the cruder load and energy criteria. Nevertheless, a study of spectra is valuable for a better understanding of shock phenomena, and, in certain cases, it may prove feasible to specify shock inputs more precisely. Figures 13 and 14 show spectra derived from motion and pressure, as explained in 9110-7-f.

Effects of design changes.–If a point is plotted to represent an assumed structure, constituting the spring of a single-degree-of-freedom system, its position relative to the shock design spectrum can be altered by various design changes. Figure 12 illustrates this for data calculated in 9110-7-c. More specifically, figure 15 shows the effects of varying the following:

1. Allowable stress. If the use of a higher strength material or a reduced slenderness ratio permits increased stresses throughout the structure, the point representing the design moves vertically upward. P/W and d are proportional to stress, and Q/W to stress squared. There is no change in frequency. (It should be remembered that both P and Q correspond to a stress within the elastic limit, and represent the structure rather than the design criteria.)

2. Area of member. An increase in area, without change in radius of gyration, moves the point diagonally, upward and to the right. P/W and Q/W increase in proportion to the area; frequency changes with its square root, while deflection remains constant.

3. Depth of member in bending. Figure 15(a) shows the change which would result from doubling the depth of the cross section, retaining the same area and shape otherwise. The point moves horizontally to the right. Q/W is unchanged; P/W and frequency increase; deflection, for the given stress level, is reduced.

4. Radius of gyration of member in bending. If an I section is substituted for a rectangular one of the same depth and area, the effect is about the same as that of doubling the area. Stiffness, strength, and energy capacity, all vary with r^2. Permissible deflection is fixed by the value of c.

5. Length of member. If the length of a member is doubled, its energy capacity is also doubled, given the same cross section and similar patterns of stress and strain. For an axially loaded member, P/W is unaffected, provided there is no instability problem. For a beam whose length is doubled, however, loading must be reduced by ½ while deflection is increased by a factor of 4.

The above relationships will suggest means of designing to suit a given shock spectrum while giving due consideration to desired structural properties. For instance, if members are stressed in bending, they will be less efficient than axially loaded members, with respect to both strength and energy. They may, however, be able to deflect far enough to withstand the shock, while transmitting a greatly reduced acceleration to the equipment. For a given pattern of stresses, weight is proportional to Q, and is therefore minimized by meeting the spectrum at the lowest possible point. Short, stiff members are likely to be lightest, but sometimes advantage can be taken of the flexibility of slender beams.

9110-7-f. Numerical Integration for Calculation of Forces and Deformations Resulting from Shock Loadings.

The following method of numerical integration can be used to calculate the effects of shock loadings, defined as either motions or external pressures. This method can be applied to input and response curves of any shape. Moreover, it does not require knowledge of advanced mathematics or reliance upon predetermined charts. Although equations and charts can be extremely useful for typical conditions, calculations such as those described below give the designer a "feel" for what happens under shock loadings which he does not get from consideration of only static equivalents.

The time history of the shock input and resulting deformations is divided into a number of discrete time intervals. These intervals must be small, but in many design problems the number which must be considered is not great.

Each mass in the system is acted upon by forces, which may include external pressure, spring forces, and damping. The net force accelerates the mass.

Spring forces are approximated on the basis of the relative positions which the masses would have at the middle of the time interval if their velocities did not change. The displacement change caused by acceleration is small compared to that caused by velocity, and, for constant acceleration, only one-fourth of the increment is effective at the middle of the time interval. Hence, this approximation causes no significant error if the time interval is suitably small.

Damping forces are based on velocities at the beginning of the time interval, together with the accelerations resulting from external forces, spring forces, and the damping forces themselves. The variations in damping forces resulting from the changing velocities can sometimes be taken into account by merely modifying the factors which would otherwise be applied to obtain the accelerations.

The method described here avoids the trial-and-error procedures which are sometimes used to allow for variations in the accelerations over the time interval. The effects of these variations become less important as the interval is decreased, and a high degree of accuracy can be obtained, theoretically, by using a very short interval. Figure 16 indicates how computational errors may be affected by choice of the time interval. For most engineering applications, inputs, spring and damping constants, and response criteria are not known with sufficient precision to justify elaborate calculations. Simplicity has been the primary objective in devising the method of analysis.

If a mass is acted upon by a force, it is accelerated by an amount, $a = P/m = (P/W)g$. If the force is not sensibly changed during a small time interval, h, displacement of the mass changes by $(v_0 + ah/2)h$. (See figure 17.) During the first half of the next time interval, displacement changes by $(v_0 + ah)(h/2)$, if further acceleration is neglected. Displacement at the middle of the first time interval, u_0, includes the effect of velocity during the first half of that interval $(v_0h/2)$, but does not include the effect of acceleration. Therefore, displacement at the middle of the second interval is, approximately, $u = u_0 + vh$, where the new velocity, $v = v_0 + ah$. The displacement increments, vh, are obtained by adding successive values of ah^2. Acceleration may vary considerably during the interval without causing significant errors, provided the approximate mean value is used and the interval is small compared to the natural periods of the system.

Spring forces are proportional to relative displacements until the elastic limit is exceeded. For purely plastic action, force is constant as long as deformation is increasing.

For viscous damping, force is proportional to relative velocity. Consider the mass of figure 18, attached to a fixed base and acted upon by force P. If velocity were constant the damping acceleration, a_C, would be $-cv/m$. If acceleration were constant, damping force would change linearly. The average value of a_C would then be $-c(v + ah/2)/m$. Because of the damping, the acceleration caused by P is reduced to approximately $P/m(1 - ch/2m)$. Hence, the accelerations of the mass, including that from damping based on initial velocity, are multiplied by the factor, $(1 - ch/2m)$.

In general, short time intervals and simple calculations are preferable to longer intervals and the more elaborate formulas which allow for variables of secondary importance.

EXAMPLES

Elastic-plastic response of a simple system subject to a step-velocity change.—An item of equipment weighing 2.5 k. is supported by HTS structure having stiffness, K = 555 k./in. Stress under load equal to weight is 1.5 k.s.i. At yield stress, the load is then

$$P_E = (45/1.5) \times 2.5 = 75 \text{ k.},$$

and deflection is

$$(u_B - u)_E = 75/555 = 0.135 \text{ in.}$$

As deflection increases beyond this point, the acceleration of the equipment is assumed to be constant. Vibration frequency (within the elastic range) is

$$n = 3.13\sqrt{K/W} = 3.13\sqrt{555/2.5} = 46.6 \text{ c.p.s.}$$

A time interval of 2 milliseconds gives

$$nh = 46.6 \times 0.002 = 0.0932,$$

which is small enough for accuracy.

In the table below, columns 1 and 2 define the given shock motion. Column 5 gives values of ah^2 for the mass, which is acted upon only by the spring force. Up to the point of yielding,

$$ah^2 = (u_B - u)(Kg/W)h^2 = (u_B - u)(555 \times 386/2.5)(0.0002)^2 = (u_B - u) \times 0.343.$$

At yield,

$$0.135 \times 0.343 = 0.046 \text{ in.}$$

If damping were present an appropriate arrangement of data would be: t, $(u_B - u)$, $a_K h^2$, $a_C h^2$, vh, $v_B h$ (a constant in this example), and $(v_B h - vh)$.

The subscript, o, after a column number indicates the preceding value in the column. Thus the third value in column 3 (.088) is obtained by adding to the second value (.021), the second value from column 6 (.067).

The table is continued until the maximum deflection has been reached. When deflection begins to decrease, acceleration would be reduced, but this phase of the response is of no special interest.

SIMPLE SYSTEM; STEP VELOCITY CHANGE

Time, t (Mean) msec. ①	Displacements – in.			Displacement Changes – in.	
	Base, u_B ②	Equipment, u ③ = ③$_o$ + ⑥$_o$	Rel., $u_B - u$ ④ = ② - ③	Increment, ah^2 ⑤ = ④ × (0.343)	Total vh ⑥ = ⑥$_o$ + ⑤
1	.060	0	.060	.021	.021
3	.180	.021	.159	.046	.067
5	.300	.088	.212	.046	.113
7	.420	.201	.219	.046	.159
Totals	0.960	0.310	0.650	0.159	0.360
9	.540	.360	0.180		

Checking.—Much of the numerical work can be checked by totaling the columns. The combined sums of 3 and 4 must equal the sum of 2. (0.310 + 0.650 = 0.960). Also, the sum of column 5 must agree with the last value in 6, and the sum of column 6 gives the next value for column 3 (0.360). If the response were elastic only, the column 5 total would also check the column 4 total times 0.343.

Extrapolation for peak values.—Figure 19 shows how the calculated velocities compare with theoretical values. The curve of relative velocity $(v_B - v)$ is a straight line, passing through zero at about 6.3 msec., at which time relative displacement is maximum. While the relative displacement computed for 7 msec. is sufficiently accurate for most design purposes, a simple extrapolation can be made to obtain the peak value. If u_o is the largest calculated value and v_1h and v_2h represent the changes just before and after:

$$u_{max} = u_o + \frac{(v_1h + v_2h)^2}{8(v_1h - v_2h)} = 0.219 + \frac{(0.007 - 0.039)^2}{8(0.007 + 0.039)} = 0.222 \text{ in.}$$

Simple system with other motion inputs.—A step velocity change can be approached, but never fully realized since it implies infinite acceleration. If the input velocity is a sine curve from 0 to the maximum and then constant, corresponding design spectra can be established by assuming various frequencies. Figure 13 shows such a spectrum, plotted nondimensionally, for elastic response.

If the velocity corresponding to yielding of the structure is less than that indicated by a spectral curve (for elastic response), plastic deflection, d_p, $(Q - Q_E)/P_0$. If P_0 and P_E (figure 3) are nearly equal, the ratio of plastic to elastic deflection is approximately

$$\frac{d_P}{d_E} = \frac{1}{2}\left[\left(\frac{V}{V_E}\right)^2 - 1\right] = \frac{1}{2}\left(\frac{Q}{Q_E} - 1\right),$$

where V and Q are the elastic-design velocity and energy, respectively, and V_E and Q_E are the values at yield.

The calculations for response of a simple, undamped system subject to any given motion are similar to those for the step velocity change. Column 2 is modified to suit the given input data.

Two masses, one of which is subjected to an external pressure pulse.—Figure 14 shows such a system. The base, W_B, is acted upon by the pressure, which decreases uniformly from $2P_o$ to 0 in time T, and by the spring force. The attached weight, W, is accelerated by the spring force only.

If T = 0.0007 sec., W/W_B = 1.8, and the rigid-base frequency of W is 500 c.p.s., calculations for elastic response can be made as follows.

For a time interval of h = 0.0001 sec., values of ah^2 resulting from the spring force are:

$$\text{For } W,\ (Kg/W)h^2(u_B - u) = (2\pi \times 500)^2(0.0001)^2(u_B - u) = 0.0987(u_B - u).$$

$$\text{For } W_B,\ (Kg/W_B)h^2(u - u_B) = -1.8 \times 0.0987(u_B - u) = -0.1777(u_B - u).$$

These coefficients of $(u_B - u)$ are small enough to ensure accuracy with the assumed value of h.

During the first time interval, the acceleration of W_B from the external pressure is

$$(13/14)(2P_o g/W_B)h^2 = 7.17(P_o/W_B) \times 10^{-6}.$$

Succeeding values are reduced proportionately, as listed in column 5.

Two-Mass System; Triangular Pressure Pulse

Time, t (Mean) msec	Displacements in. × $(P_o/10^6W_B)$			Displacement Changes – in. × $(P_o/10^6W_B)$				
				Base			Equipment	
	u_B	u	$u_B - u$	Pgh^2/W_B	$(u_B-u)\left(\frac{Kgh^2}{W_B}\right)$	v_Bh	$(u_B-u)\left(\frac{Kgh^2}{W}\right)$	vh
①	② = ②$_o$ + ⑦$_o$	③ = ③$_o$ + ⑨$_o$	④ = ② - ③	⑤	⑥ = ④(-.1777)	⑦ = ⑦$_o$ + ⑤ + ⑥	⑧ = ④(.0987)	⑨ = ⑨$_o$ + ⑧
0.1	0	0	0	7.17	0	7.17	0	0
0.3	7.17	0	7.17	6.07	-1.27	11.97	0.71	0.71
0.5	19.14	0.71	18.43	4.96	-3.28	13.65	1.82	2.53

(Continued)

Two-Mass System; Triangular Pressure Pulse (Continued)

Time, t (Mean) msec ①	Displacements in. × ($P_o/10^6W_B$)			Displacement Changes – in. × ($P_o/10^6W_B$)				
				Base			Equipment	
	u_B ② = ②$_o$ + ⑦$_o$	u ③ = ③$_o$ + ⑨$_o$	$u_B - u$ ④ = ② - ③	$P_g h^2/W_B$ ⑤	$(u_B-u)\left(\frac{K_B h^2}{W_B}\right)$ ⑥ = ④(-.1777)	$v_B h$ ⑦ = ⑦$_o$+⑤+⑥	$(u_B-u)\left(\frac{K_B h^2}{W}\right)$ ⑧ = ④(.0987)	vh ⑨ = ⑨$_o$+⑧
0.7	32.79	3.24	29.55	3.86	-5.25	12.26	2.92	5.45
0.9	45.05	8.69	36.36	2.76	-6.46	8.56	3.59	9.04
1.1	53.61	17.73	35.88	1.65	-6.38	3.83	3.54	12.58
1.3	57.44	30.31	27.13	0.55	-4.82	-0.44	2.68	15.26
Totals	215.20	60.68	154.52	27.02	-27.46	57.00	15.26	45.57
1.5	57.00	45.57	11.43	–				

Check: 215.20 – 60.68 = 154.52; 154.52 × (–.1777) = 27.46; 154.20 × (.0987) = 15.25 = 15.26.

The calculations indicate that maximum relative elastic displacement is about $37(P_o/10^6W_B)$. The corresponding design velocity can be obtained by multiplying this value by ω, or $2\pi n$.

Figure 14 shows spectra, derived by a similar procedure, for various ratios of W/W_B, plotted non-dimensionally. Since the free-base velocity is P_oTg/W_B, the relative value for the above example is

$$\frac{37(P_o/10^6W_B)(2\pi \times 500)}{P_o(0.0007)(386)/W_B} = 0.43 .$$

This is confirmed by reference to the curves.

The same procedure can be extended to include other forces and masses. However, if many masses are involved, particularly if there are large differences in their rigid-base frequencies, a high-speed computer becomes virtually a necessity, unless the system can be reduced to simpler equivalents.

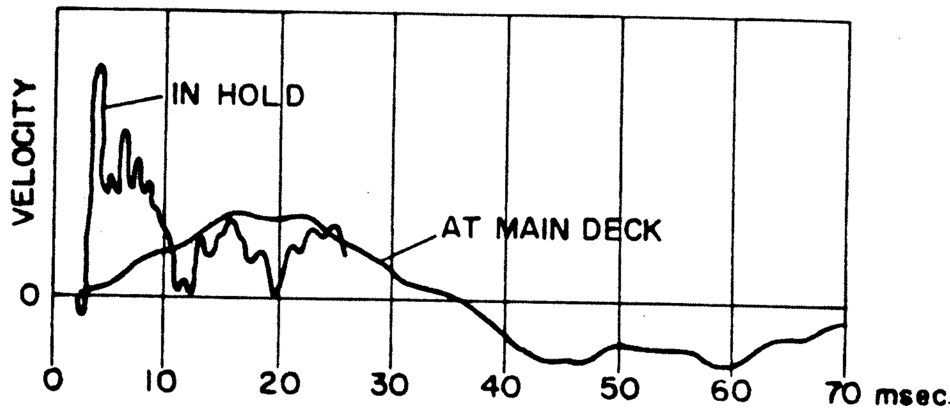


Figure 1. Vertical motion measured during shock test of destroyer.

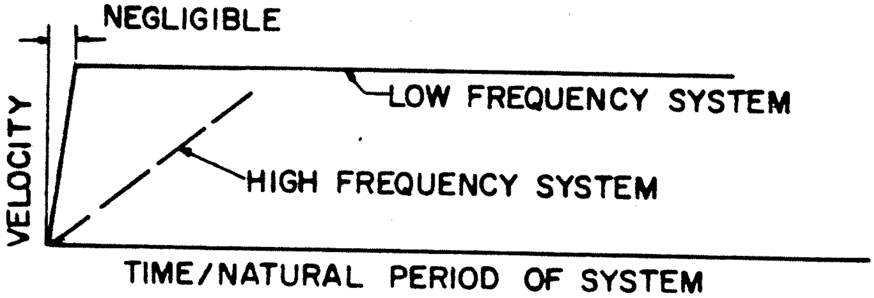


Figure 2. Motion as idealized for design.

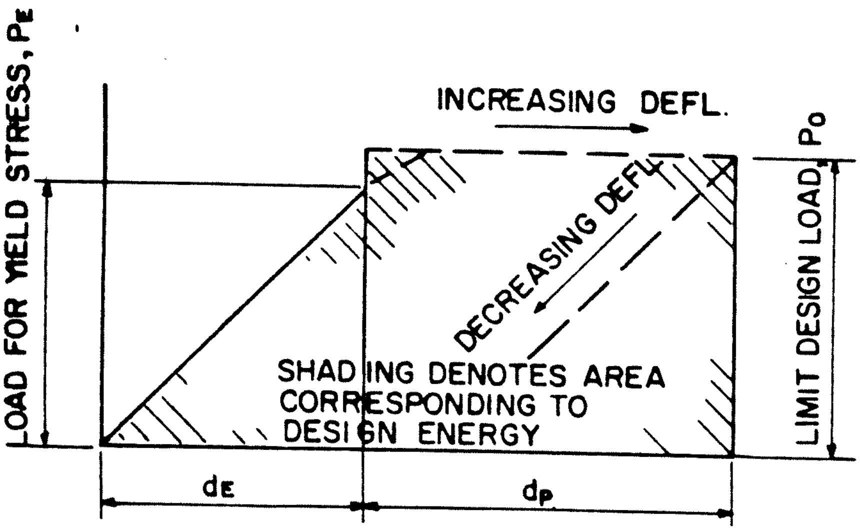


Figure 3. Load-deflection curve.

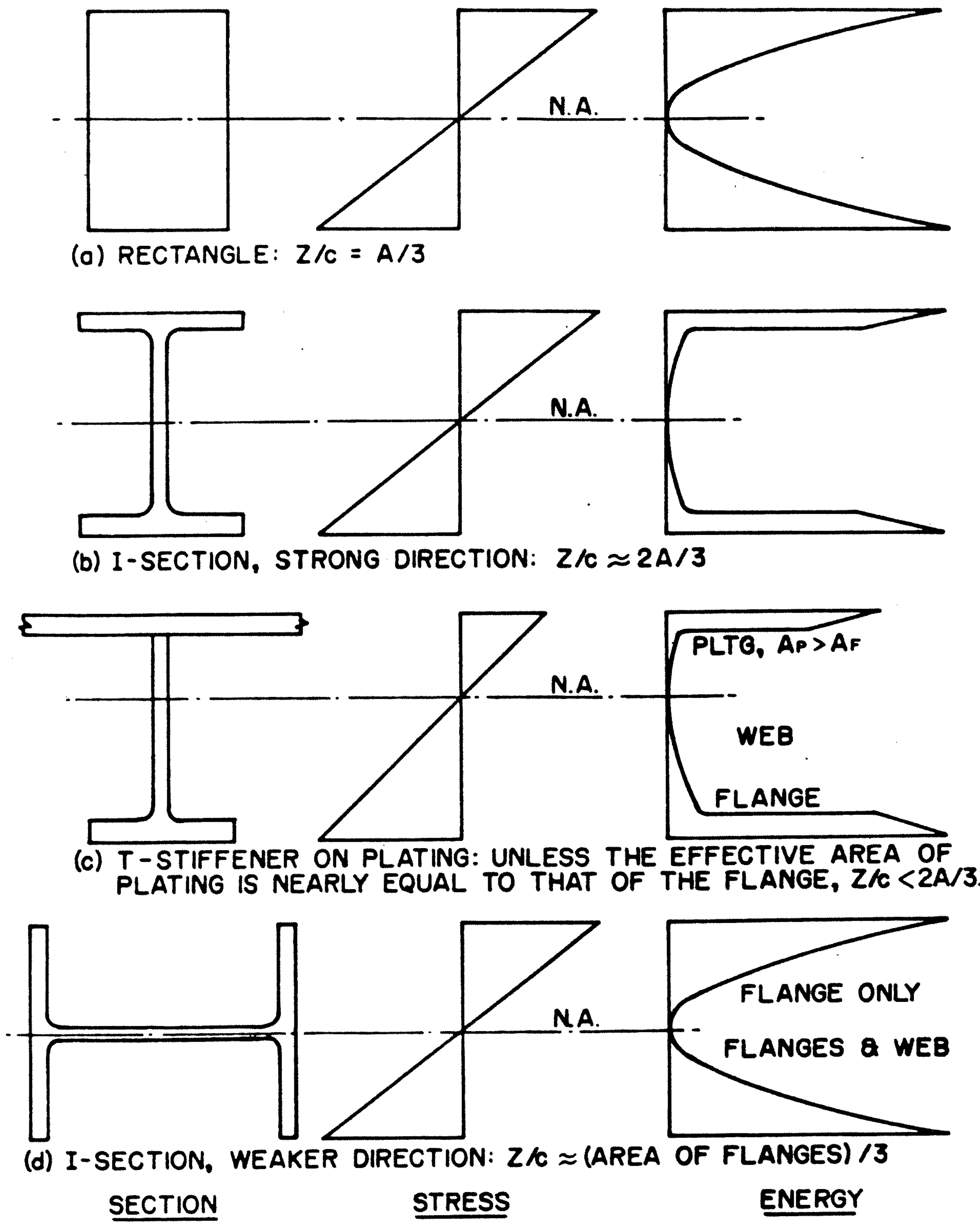


Figure 4. Stress and energy diagrams for various cross-sections in bending.

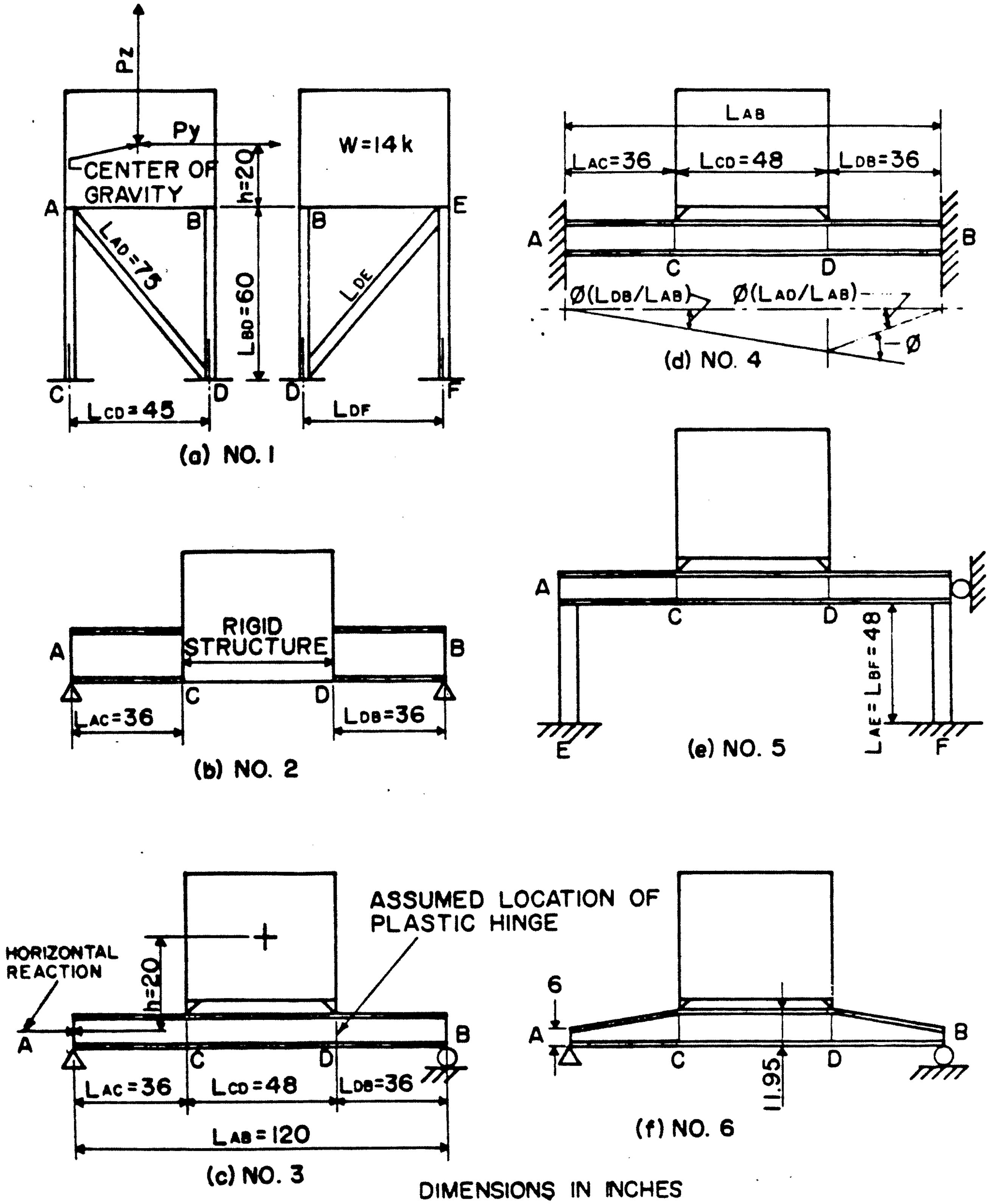


Figure 5. Foundation arrangements for single-mass system.

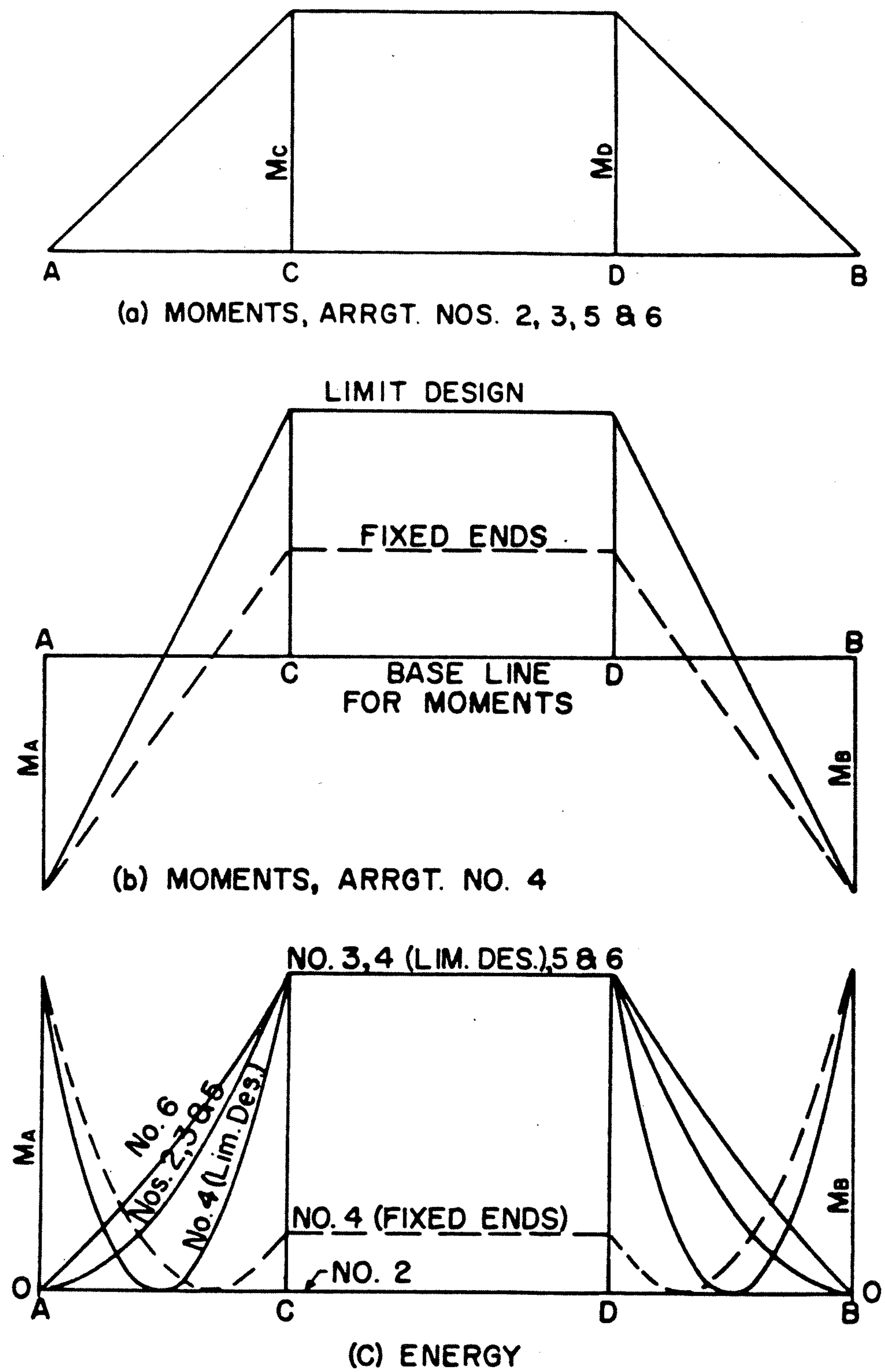
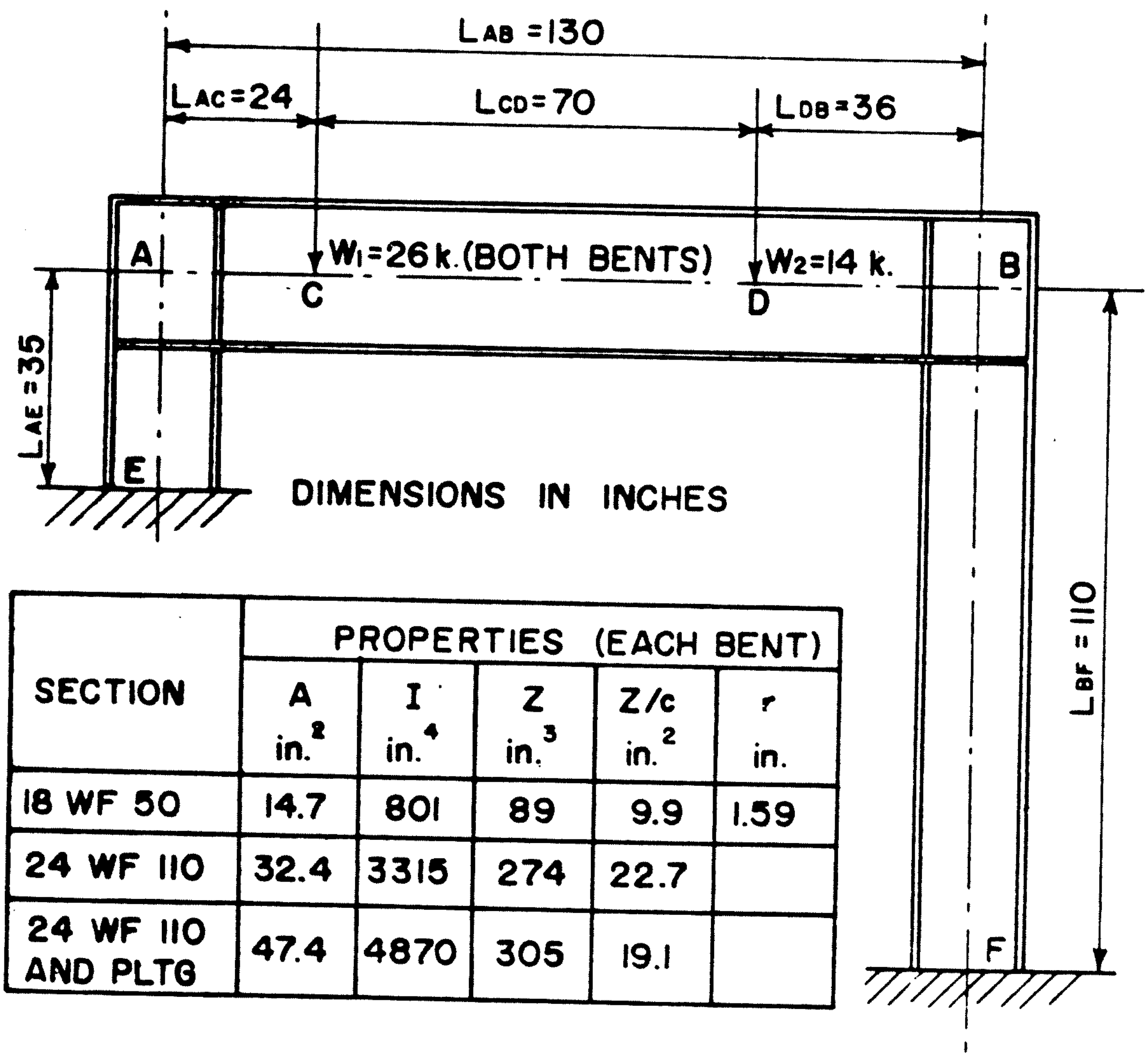
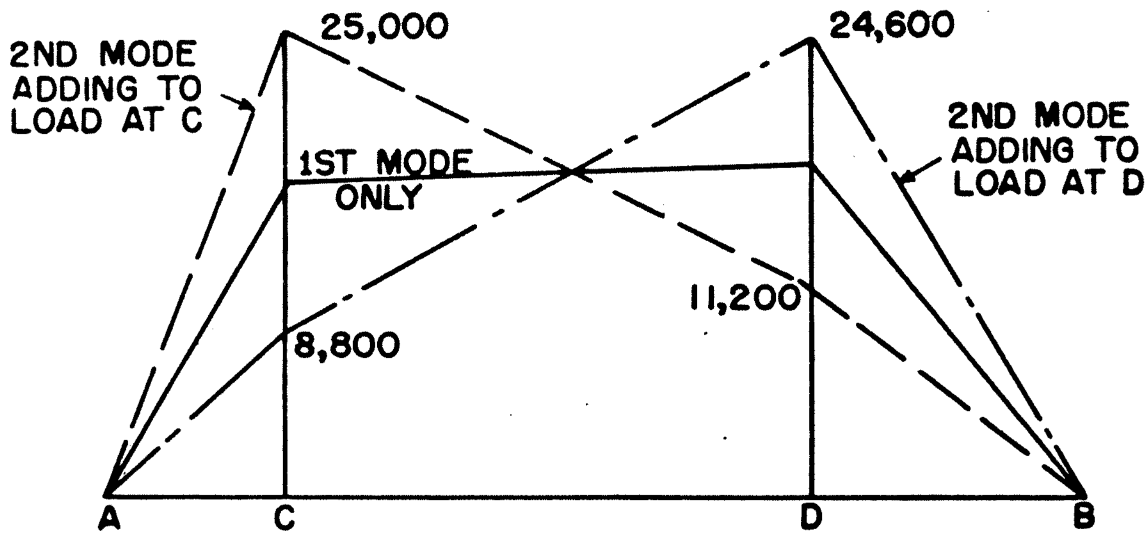


Figure 6. Moment and energy diagrams for girders shown in figure 5.

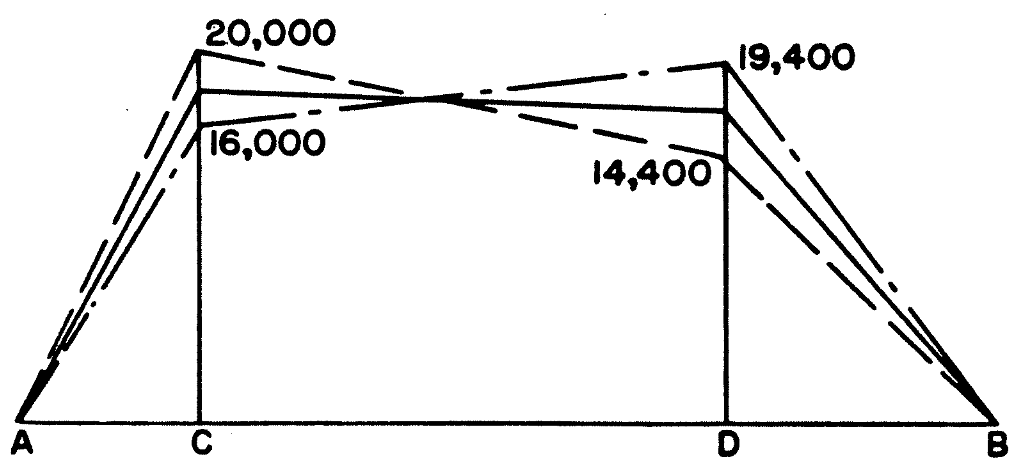


SECTION	PROPERTIES (EACH BENT)				
	A in.^{2}	I in.^{4}	Z in.^{3}	Z/c in.^{2}	r in.
18 WF 50	14.7	801	89	9.9	1.59
24 WF 110	32.4	3315	274	22.7	
24 WF 110 AND PLTG	47.4	4870	305	19.1	

Figure 7. Structural arrangement and section properties for two-mass system.



(a) UNEQUAL LEGS, AS SHOWN IN FIGURE 7.



(b) BOTH LEGS 72.5 INCHES LONG.

Figure 8. Bending moments in horizontal girders of two-mass system under vertical shock loading with pin joints at A and B. Shear deflections are neglected and I-sections only are considered.

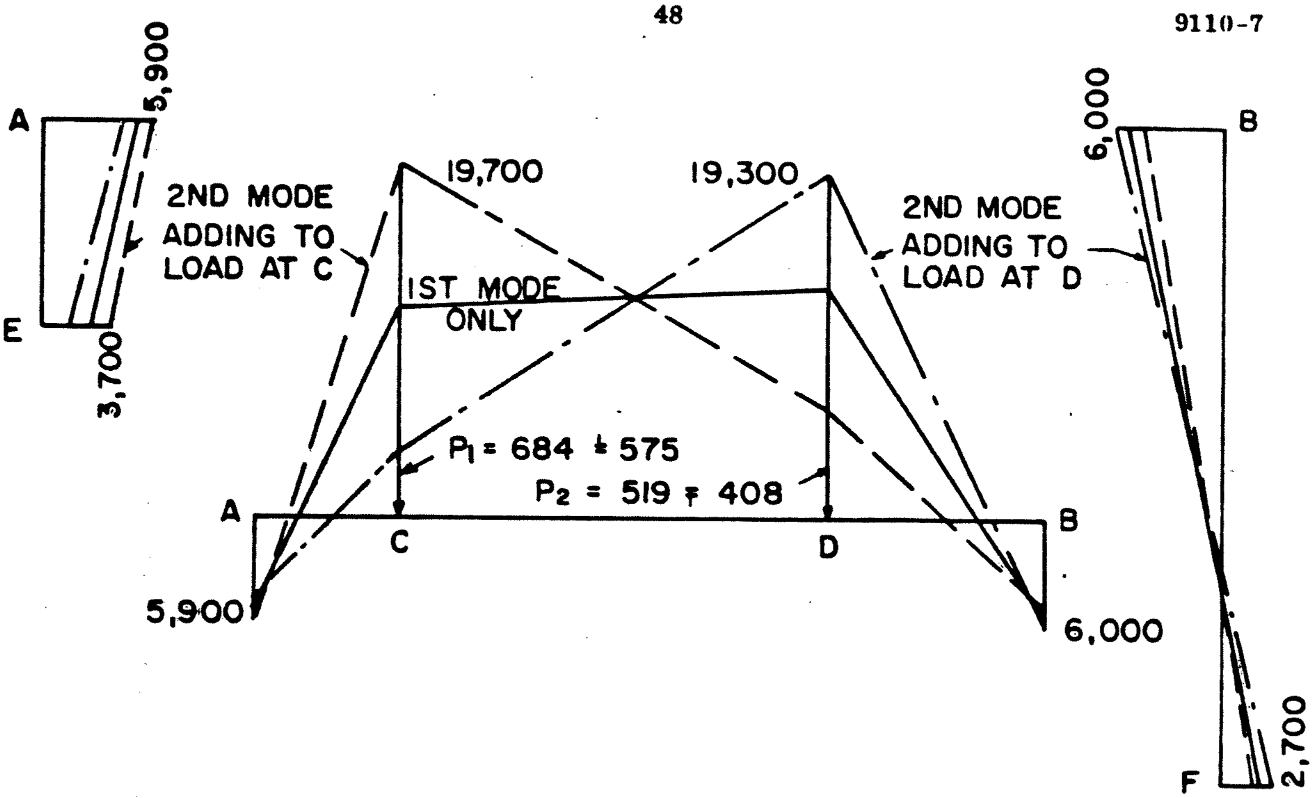


Figure 9. Bending moments in members, with rigid joints at A and B. Assumed conditions are otherwise the same as for figure 8(a).

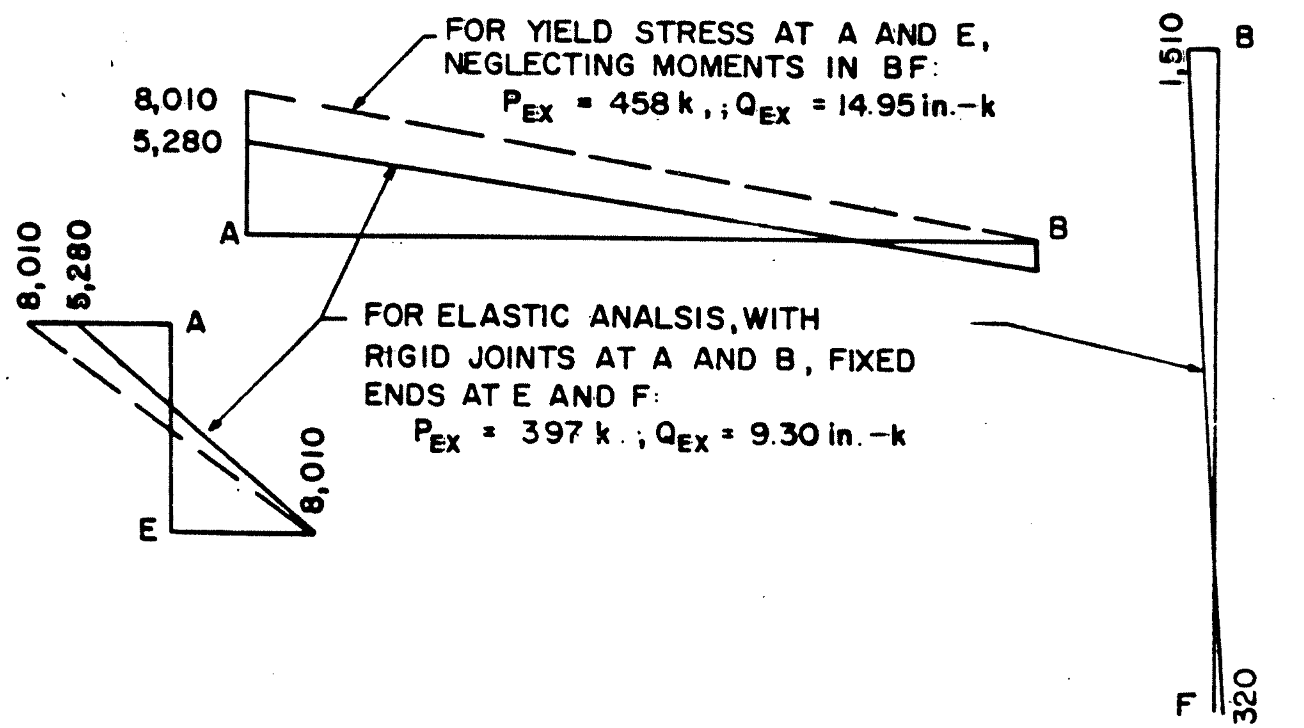


Figure 10. Bending moments under longitudinal loadings (along AB). The structural arrangement is shown in figure 7.

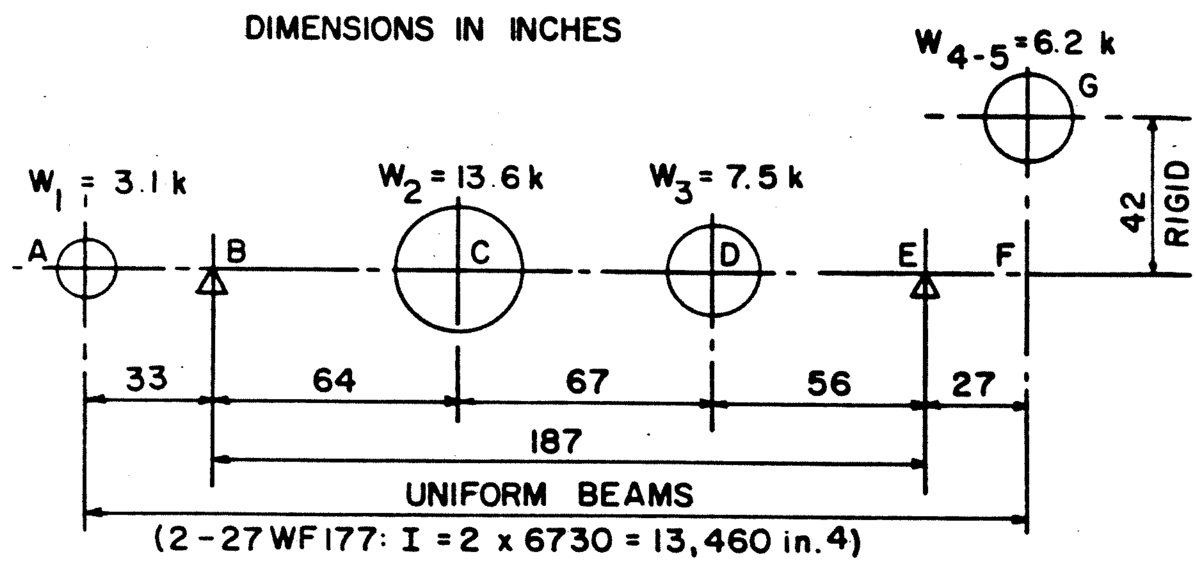


Figure 11. Arrangement of four-mass system having five degrees of freedom. The only deflections considered are those due to bending of the horizontal beams.

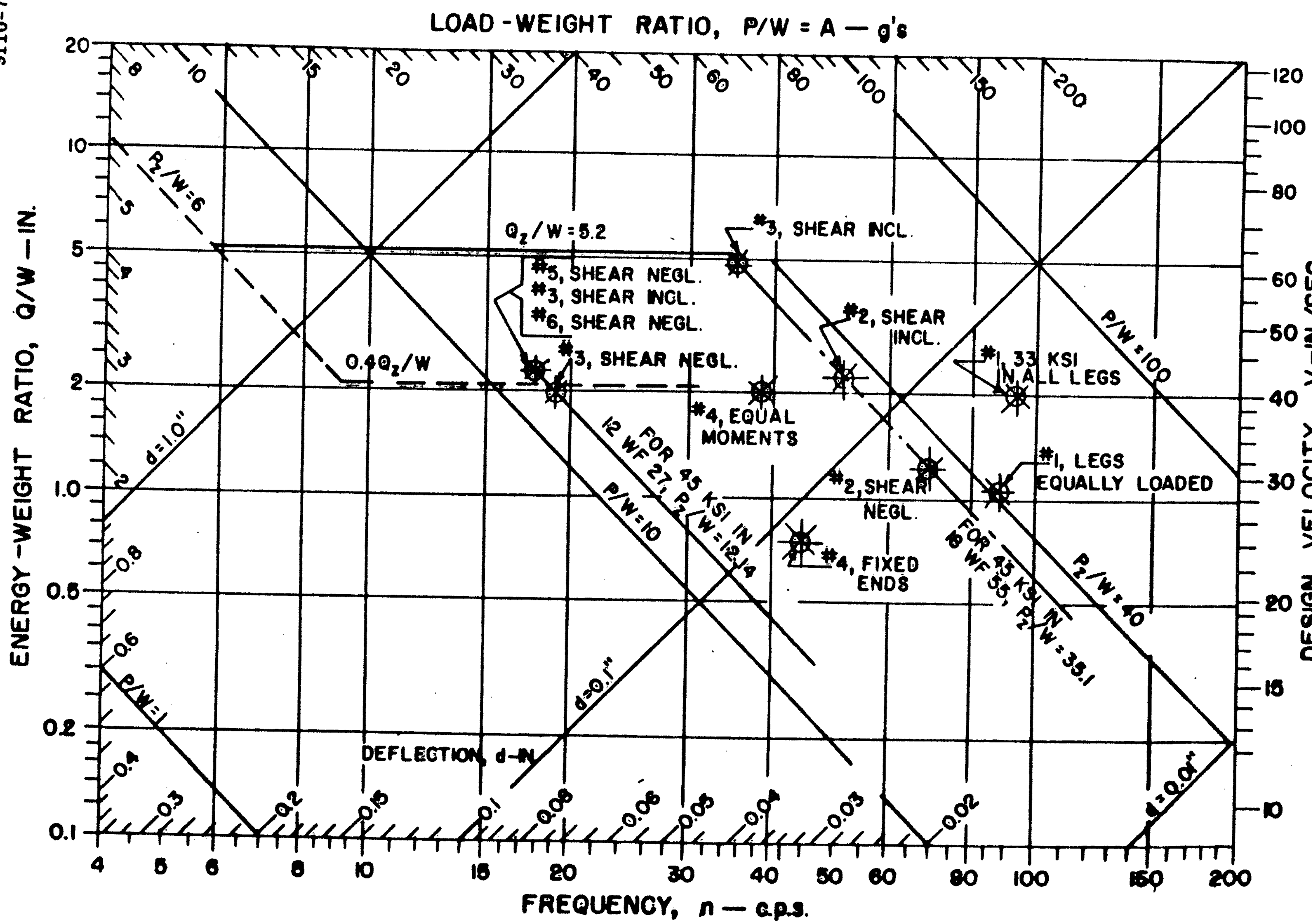


Figure 12. Shock design spectrum for examples of 9110-7-c. Vertical loading, W = 14 kips. Arrangement numbers refer to figure 5.

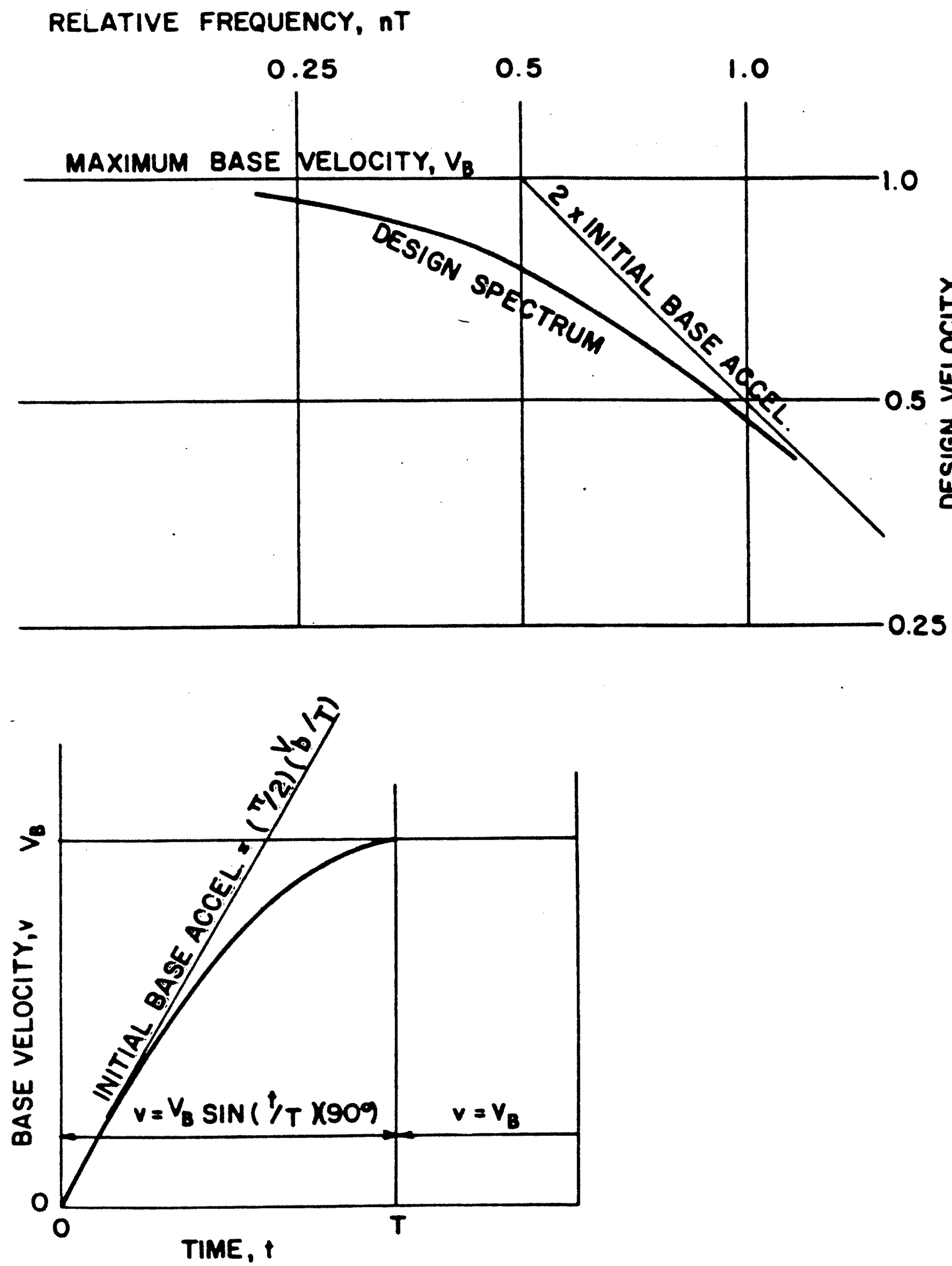
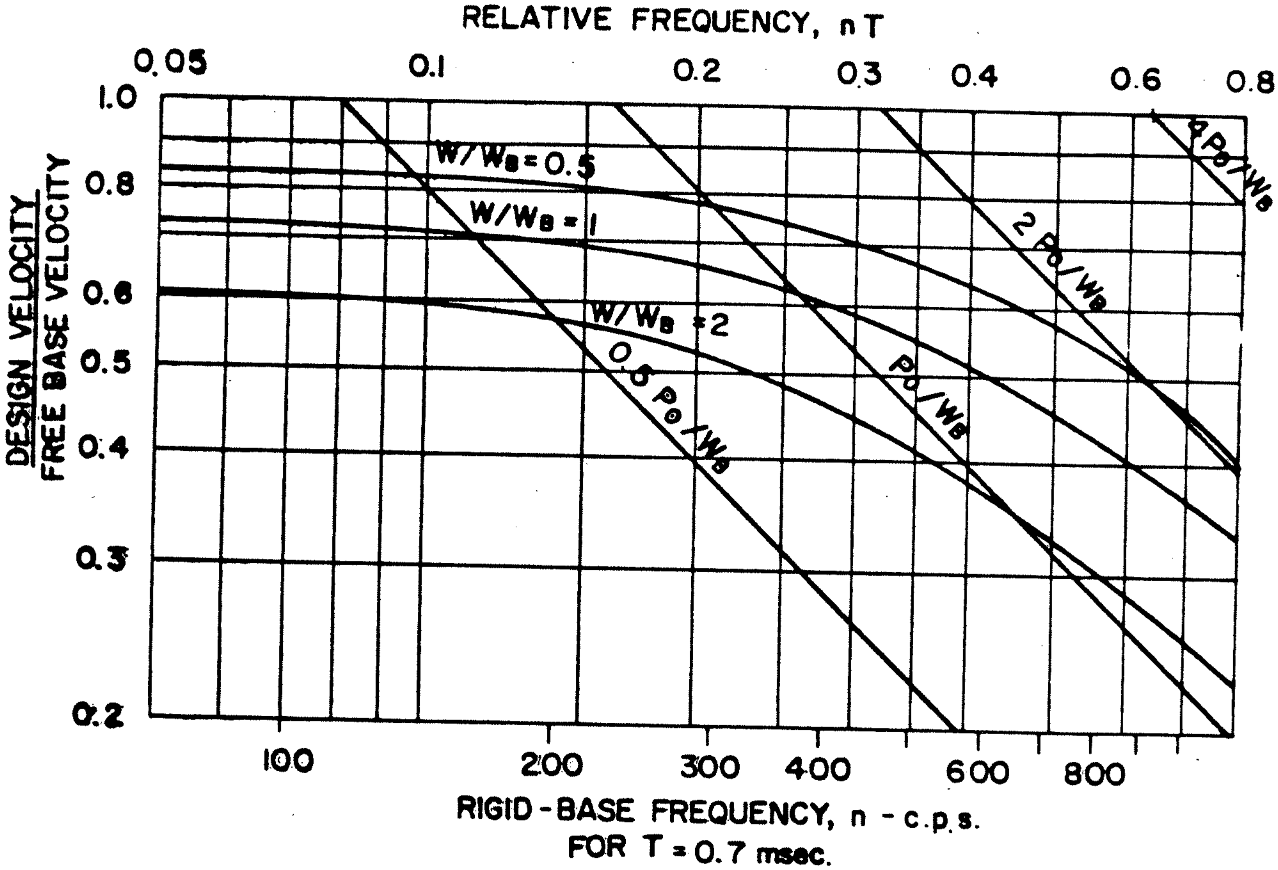


Figure 13. Spectrum for quarter-sine-wave base-velocity curve.



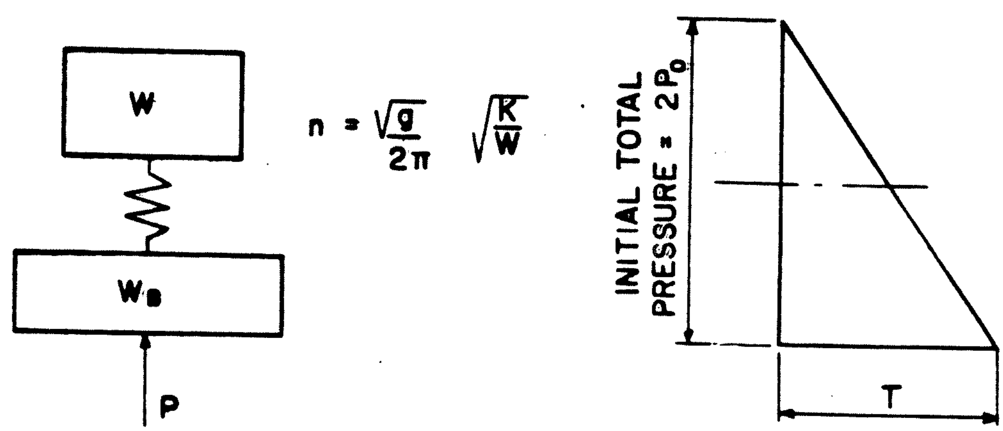
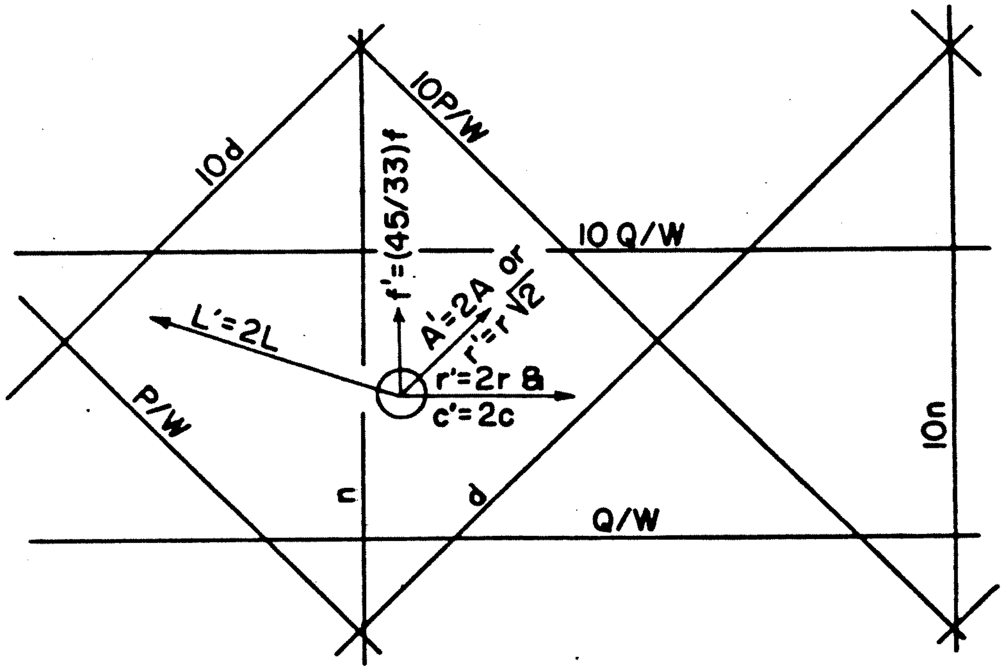
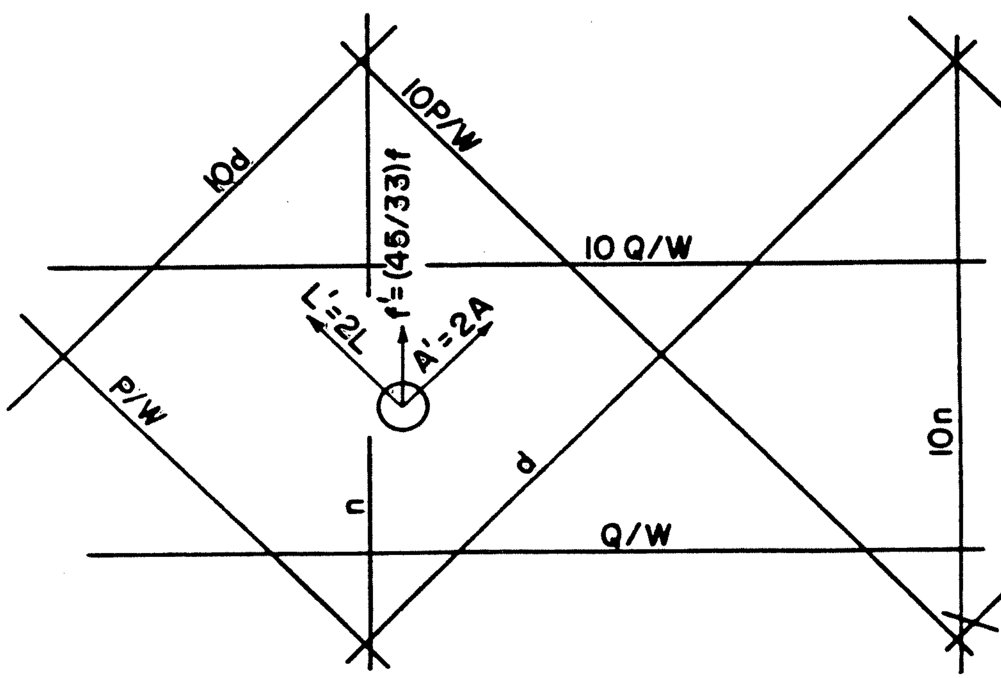


Figure 14. Spectra for triangular external pressure pulse. Free-base velocity, $V_B = (P_oT)\, g/W_B$, where P_oT is the area under the pressure-time curve. If nT is small, design velocity is nearly $V_B/\sqrt{1 + W/W_B}$



(a) BEAM FOUNDATION



(b) UNIFORMLY STRESSED STRUCTURE

Figure 15. Effects of design changes relative to spectra.

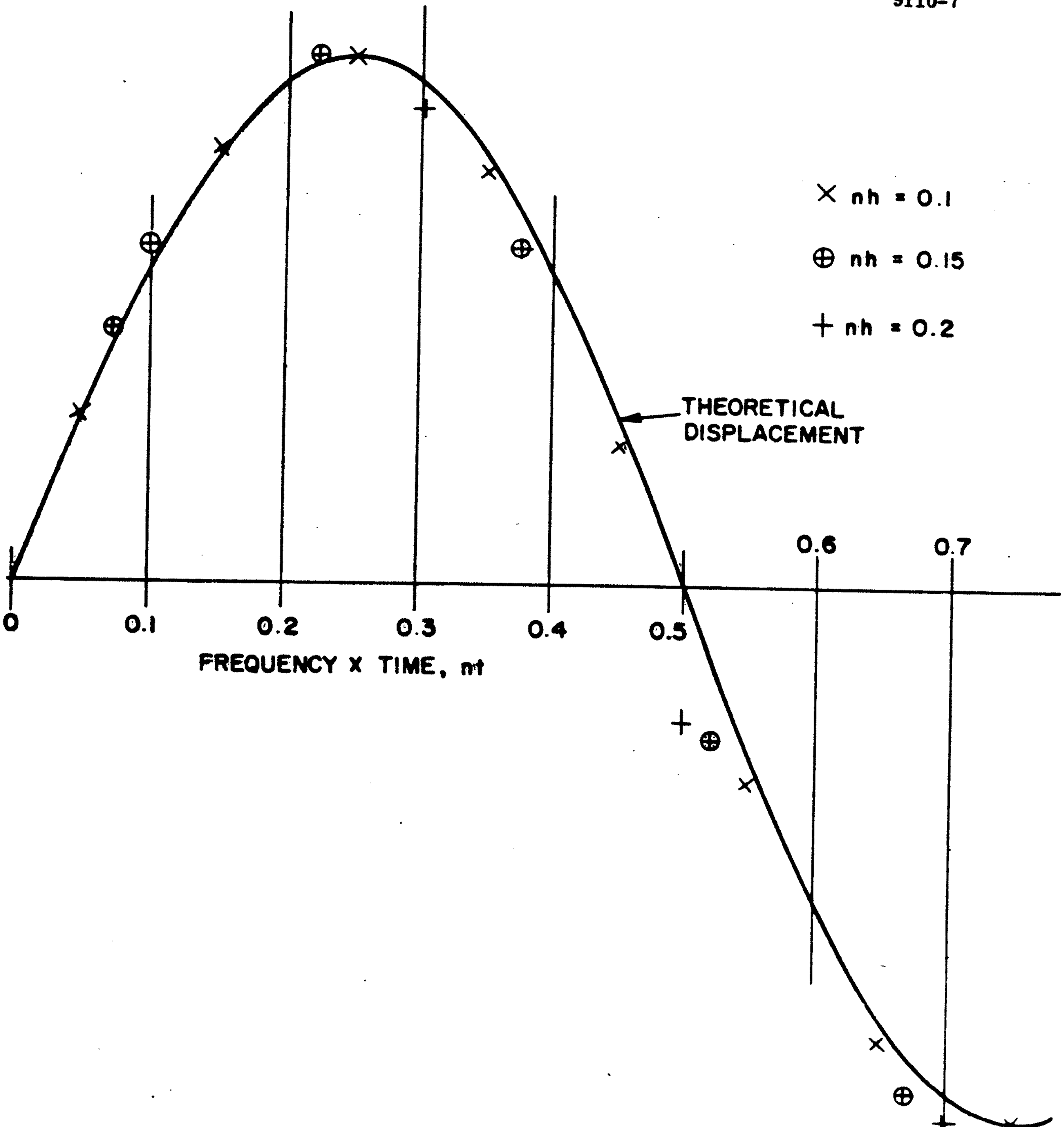


Figure 16. Relative displacements for a simple, undamped, elastic system with its base subjected to a step velocity change. For the larger time intervals there is an appreciable error in frequency. The ratio of the true frequency to the apparent frequency is $\sin(\omega h/2)/\omega h/2$ $\sin(180\ nh)^{\circ}/\pi\ nh$.

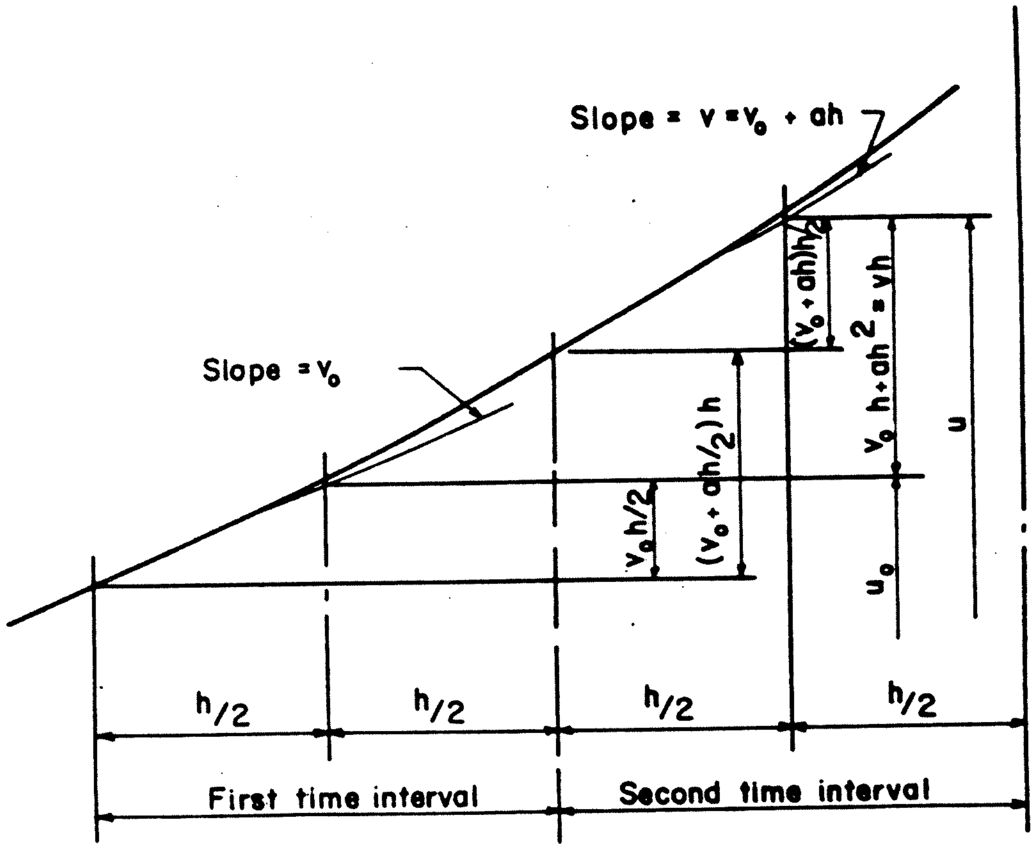


Figure 17. Displacement of mass, given an acceleration, a, for a time interval, h.

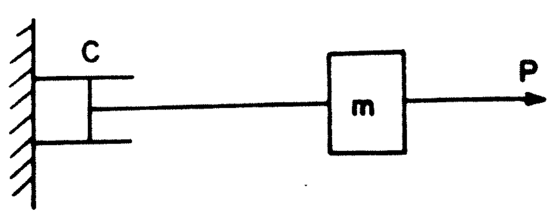


Figure 18. Mass whose motion is influenced by viscous damping.

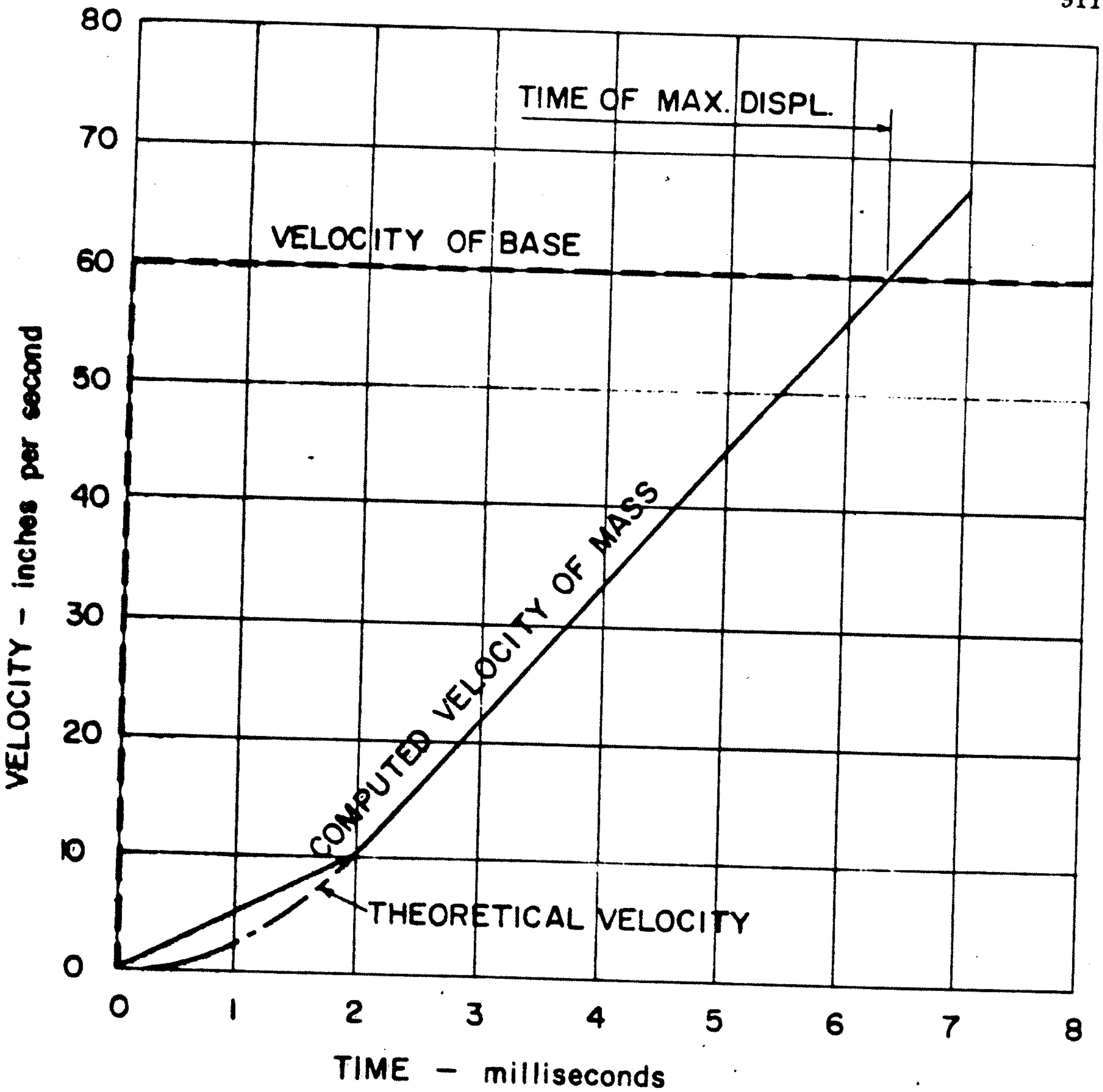


Figure 19. Velocity-time curve for step velocity change. Note that tabulated displacements correspond to the middles of the intervals, tabulated velocities to their ends. The theoretically exact curve of motion is a versed sine curve from 0 to 2.4 msec.